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Riccati Observers for the non-stationary PnP problem

Tarek Hamel and Claude Samson

Abstract—This paper revisits the problem of estimating the pose (position and orientation) of a body in 3D space with respect to (w.r.t.) an inertial frame by using i) the knowledge of source points positions in the inertial frame, ii) the measurements of the body velocity, either in the body frame or in the inertial frame, and iii) source points bearing measurements performed in the body frame. An important difference with the much studied *static* Perspective-n-Point (PnP) problem addressed with iterative algorithms is that body motion is not only allowed but also used as a source of information that improves the estimation possibilities. With respect to the probabilistic framework commonly used in other studies that develop Extended Kalman filter (EKF) solutions, the deterministic approach here adopted is better suited to point out the observability conditions, that involve the number and disposition of the source points in combination with body motion characteristics, under which the proposed observers ensure robust estimation of the body pose. These observers are here named Riccati observers because of the instrumental role played by the Continuous Riccati equation (CRE) in the design of the observers and in the Lyapunov stability and convergence analysis that we develop independently of the well-known complementary (either deterministic or probabilistic) optimality properties associated with Kalman filtering. The set of these observers also encompasses Extended Kalman filter solutions. Another contribution of the present study is to show the importance of using body motion to improve the observers performance and, when this is possible, of measuring the body translational velocity in the inertial frame rather than in the body frame in order to remove the constraint of knowing the positions of the source points in the inertial frame. This latter issue is the link that connects the problem of body pose estimation with single source point bearing measurements and the Simultaneous Localization and Mapping (SLAM) problem in Robotics.

Index Terms—Observers for nonlinear systems, observability, Perspective-n-Point problem, Riccati equation

I. INTRODUCTION

Body pose estimation from source points bearing measurements is well exemplified by the problem of estimating the pose of a monocular perspective camera from projected positions of observed source points measured in the camera images. When the attitude (orientation) of the camera is measured or estimated by other means, the problem reduces to the one of estimating the position of the camera. By contrast with the complete pose estimation problem, this simpler sub-problem can be exactly linearised and yield observers that are *globally* exponentially stable under adequate observability

conditions, see for instance [1]. Attitude estimation introduces another level of complexity in relation to the structure of the compact Lie group of rotations $SO(3)$ involved in the larger group $SE(3)$ of 3D rigid transformations. In particular exact linearisation of motion equations on these groups is not possible and, even more annoying, globally convex penalty functions needed to the existence of gradient-based observers achieving global asymptotic stability of zero estimation errors do not exist. The design methodology here adopted relies on approximate linearisation in the spirit of Extended Kalman filtering (EKF), except that it is derived in a deterministic framework that allows for a clear exposition of observability conditions under which the proposed iterative observers are endowed with robust stability and convergence properties. For this reason, and also because of the instrumental role played by the Continuous Riccati Equation (CRE) in both the observers equations and the definition of the associated Lyapunov cost functions used for stability and convergence analysis, we propose the generic name of Riccati observers to refer to the class of observers here studied and whose expressions encompass EKF solutions. This point of view does not hinder a complementary probabilistic modelling of measurement perturbations that may be useful to efficiently tune the observers parameters by application of Kalman filtering rules.

Following the excellent survey [2], the perspective pose estimation problem with three source points (i.e. the minimal number needed to solve the problem in the static case when the camera and the source points are motionless) was first solved via algebraic calculations (the direct solution method) by a German mathematician in 1841 [3]. His solution consists in first determining the distances from the optic centre to the source points so as to obtain an estimation of the optic centre position expressed in the camera frame and, in a second stage, in determining the camera attitude. It was subsequently, and until recently, refined by photogrammetrists and computer vision specialists [4], [5], with complementary analyses concerning the number of poses satisfying the perspective projection equations associated with the problem and the numerical stability of the proposed solutions (study of singular solutions). This latter problem points out in particular the existence of the so-called *danger cylinder* and the numerical instability of the solutions when the camera optic centre is located on this cylinder (whose equation is simply obtained by zeroing the determinant of the Jacobian matrix associated with the equations [2], [6]). The extension of these analyses to four and five source points has also given rise to several publications [4], [7].

Iterative algorithms based on gradient search [8], [9] are often motivated by the observation of more than four

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source points with bearing measurement errors and possible outliers (resulting from incorrect point correspondences), with the aim of determining a pose estimate that minimizes a quadratic average error index, itself derived from the perspective projection equations. These methods are local by nature and are not directly concerned with the number of solutions to the perspective projection equations, provided that this number is finite to ensure that the error index function admits isolated (local) minima. However, the domain of convergence to a "good" solution, as well as the rate of convergence to such a solution, are related to the same regularity conditions as those associated with the stability of algebraic resolution methods. In particular, the convexity of the index function in the vicinity of the body position, which is needed to ensure robust convergence of any gradient-based algorithm to this point, requires that the rank of the Jacobian matrix associated with the perspective projection equations is equal to three. With three source points, this condition is thus again not satisfied when the optic centre is located on the danger cylinder. The analysis developed in the present paper also accounts for this type of problem via a different approach.

The approach taken in this paper may be viewed as a prolongation of the iterative approach to the more general and comparatively little studied non-stationary case involving possible body motion and on-line body pose estimation from measurements acquired over time. Related works in this direction, sharing features with the observer design methodology proposed herein, can be found in [10], [11] and references therein. We here use the Automatic Control notion of *uniform observability* [12] to characterize the aforementioned regularity conditions and show that the explicit use of measured velocity in complementation to bearing measurements can weaken the conditions under which effective and robust pose estimation can be achieved. The relation between uniform observability and well-posedness of the Riccati equation involved in Kalman filtering is also recalled. In this respect and in the same way as the existence of solutions to the perspective projection equations does not systematically imply that the solution is numerically stable, weak observability is not sufficient to derive fast converging and robust observers. This explains the accent put in the paper on the stronger property of uniform observability.

The paper is organized as follows. Notation used throughout the paper and recalls of basic definitions related to the CRE and conditions under which a matrix-solution to a CRE associated with a time-varying linear system and its inverse are uniformly bounded are given in Section II. In Section III a generic dynamic system verifying a set of structural properties, complemented with a pre-observer system involving a CRE, is defined prior to stating conditions, reported in a theorem, under which local exponential stability of zero estimation errors is achieved for this dual system. This theorem directly applies to a number of systems evolving on $SE(3)$ with associated Riccati (EKF-like) state observers. Its application to the estimation of a body pose from velocity and source points bearing measurements is detailed in Section IV by considering several minimal parametrizations of $SE(3)$. By virtue of the theorem stated in the previous

section, the Riccati observers so obtained share, despite their differences, the same local stability and convergence properties. The observers are first derived in the case where the body translational velocity is measured in the body frame, then in the case where this velocity is measured in the inertial frame w.r.t. which the source points positions are known. Non-uniform observability –translated in terms of source points numbers, singular geometric localizations, and body motion– that jeopardizes the performance of the observers is analysed in details in both cases. This analysis provides an alternate means to recover static singular configurations well-known in photogrammetry such as the danger cylinder, in the case of three source points, and horopter curves, in the case of four and more source points. It also points out how body motion can be useful to overcome these singularities and, perhaps more importantly, it shows that the measurement of the body translational velocity in the inertial frame combined with bearing measurement of a single source point is generically sufficient to estimate the body pose. Simulations illustrating some aspects of this analysis are reported in Section V. A short comparative analysis of stereo vs. monocular vision in terms of observability is carried out in Section VI and concluding remarks, further pointing out the connection between the single source point case and the Simultaneous Localization and Mapping (SLAM) problem in Robotics, are offered in Section VII. Finally, an extension of the approach to the case when the translational and angular body velocities are corrupted by constant additive biases is sketched out in the Appendix.

II. NOTATION AND DEFINITIONS

- $|x|$ is the Euclidean norm of the vector $x \in \mathbb{R}^n$, and $|A|$ with A a real matrix denotes the usual corresponding matrix norm.
- x^\top is the transpose of the vector x , and A^\top is the transpose of the matrix A .
- $\mathcal{B}_1^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is the closed unit ball in \mathbb{R}^n .
- $S_p^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ is the n -dimensional sphere of radius equal to one.
- S_+^n is the set of symmetric non-negative semidefinite matrices of dimension $(n \times n)$.
- $0_{m \times n}$ is the null matrix with m lines and n columns.
- 0_n and I_n are respectively the null matrix and the identity matrix of dimension $n \times n$.
- $S(x)$ is the skew-symmetric matrix associated with the cross-product in \mathbb{R}^3 , i.e. $S(x)y = x \times y$, $\forall x \in \mathbb{R}^3$, $\forall y \in \mathbb{R}^3$.
- $\Pi_x := I_3 - xx^\top$ with $x \in S_p^2$ is the orthogonal projection operator in \mathbb{R}^3 onto the two-dimensional vector subspace orthogonal to x .
- With f denoting a vector-valued function depending on the two variables x and y , and on the time variable t , we write $f = O(|x|^{k_1}|y|^{k_2})$ with $k_1 \geq 0$ and $k_2 \geq 0$ if $\forall t : |f(x, y, t)|/(|x|^{k_1}|y|^{k_2}) \leq \gamma < \infty$ in the neighbourhood of $(x = 0, y = 0)$. If f depends only on x and t then we write $f(x, t) = O(|x|^k)$ if $\forall t : |f(x, t)|/|x|^k \leq \gamma < \infty$ in the neighbourhood of $x = 0$.

Given continuous $n \times n$ -dimensional matrix-valued functions $A(t)$ and $V(t)$, with $V(t)$ non-negative semidefinite for all $t \in \mathbb{R}$, the *controllability Grammian associated with the pair (A, V)* is the non-negative semidefinite matrix-valued function defined by

$$W_V^A(t, t + \delta) := \frac{1}{\delta} \int_t^{t+\delta} \Phi(t, s) V(s) \Phi^\top(t, s) ds \quad (1)$$

with $\Phi(t, s)$ the transition matrix associated with $A(t)$, i.e. such that $\frac{d}{dt} \Phi(t, s) = A(t) \Phi(t, s)$ with $\Phi(t, t) = I_n$.

Given a continuous $n \times n$ -dimensional matrix-valued function $A(t)$, a continuous $m \times n$ -dimensional matrix-valued function $C(t)$, and a continuous $m \times m$ -dimensional matrix-valued function $Q(t)$, with $Q(t)$ non-negative semidefinite for all $t \in \mathbb{R}$, the *Riccati observability Grammian associated with the triplet (A, C, Q)* is the non-negative semidefinite matrix-valued function defined by

$$W_Q^{A, C}(t, t + \delta) := \frac{1}{\delta} \int_t^{t+\delta} \Phi^\top(s, t) C^\top(s) Q(s) C(s) \Phi(s, t) ds \quad (2)$$

If $A(t)$ and $C(t)$ are bounded and if there exists $\delta > 0$ and $\epsilon > 0$ such that $W_{I_n}^{A, C}(t, t + \delta) > \epsilon I_n$ for all $t > 0$, then we say that the pair (A, C) is uniformly observable. This is a short way of stating that the state of the Linear Time Varying (LTV) system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{aligned}$$

characterized by the pair (A, C) is uniformly observable in the sense that $x(t)$ can be calculated from the knowledge of $u(\cdot)$ and $y(\cdot)$ on the time interval $[t, t + \delta]$ (see [12]).

Given $A(t)$, $C(t)$, $Q(t)$ and $V(t)$ as specified previously, the Continuous Riccati Equation (CRE) associated with the set (A, C, Q, V) is

$$\dot{P} = A(t)P + PA^\top(t) - PC^\top(t)Q(t)C(t) + V(t)$$

with $P(0)$ a symmetric positive definite matrix. Provided that the matrices $A(t)$, $C(t)$, $Q(t)$ and $V(t)$ are bounded, the existence of $\delta > 0$ and $\epsilon > 0$ such that $W_V^A(t, t + \delta) > \epsilon I_n$ and $W_Q^{A, C}(t, t + \delta) > \epsilon I_n$ for all $t > 0$ then ensures i) that the solution $P(t)$ to the CRE exists on \mathbb{R}^+ and ii) the existence of positive numbers p_m and p_M such that $p_m I_n \leq P(t) \leq p_M I_n$ (see [1], for instance). In view of the previous definitions, the above mentioned properties of W_V^A and $W_Q^{A, C}$ are automatically granted when $V(t)$ and $Q(t)$ are larger than some positive definite matrix and the pair (A, C) is uniformly observable.

III. A FRAMEWORK FOR DESIGN AND ANALYSIS OF A CLASS OF OBSERVERS

Consider:

- a control system living in $\mathcal{B}_1^n \times \mathbb{R}^n$ and whose state $x = (x_1^\top, x_2^\top)^\top$, with $x_1 \in \mathcal{B}_1^n$ and $x_2 \in \mathbb{R}^n$, evolves according to an equation of the form:

$$\dot{x} = A(x_1, t)x + \begin{pmatrix} u_1 \\ u_2(t) \end{pmatrix} + O(|x_1|^2) + O(|x_1||u_1|) \quad (3)$$

with $\dim(u_1) = \dim(u_2) = n$, $A(x_1, t)$ a continuous matrix-valued function uniformly bounded w.r.t t (and thus bounded), uniformly continuous w.r.t. the variable x_1 in \mathcal{B}_1^n , and such that

$$A(x_1, t) = \begin{bmatrix} A_{1,1}(t) & 0_n \\ A_{2,1}(x_1, t) & A_{2,2}(t) \end{bmatrix}$$

It is also assumed that $u_2(t)$ is bounded and that the solutions to this system (thus including the initial condition $x(0)$) belong to a compact set \mathcal{D} independently of the control u_1 applied to the system.

- an "output" function $y : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^m$ such that $\forall (x, \hat{x}_2, t) \in (\mathcal{B}_1^n \times \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^+$:

$$y(x, \hat{x}_2, t) = C_1(x_1, \hat{x}_2, t)x_1 + C_2(x_1, \hat{x}_2, t)x_2 + O(|x_1|^2) + O(|x_1||x_2 - \hat{x}_2|) \quad (4)$$

with $C = [C_1, C_2]$ denoting a continuous matrix-valued function uniformly bounded w.r.t. the time variable t and uniformly continuous w.r.t. (x_1, \hat{x}_2) in a set containing \mathcal{D} .

- a second system interconnected with the first one and whose state $(\hat{x}_2, P) \in (\mathbb{R}^n \times \mathcal{S}_+^{2n})$ evolves according to

$$\begin{aligned} \dot{\hat{x}}_2 &= A_{2,2}(t)\hat{x}_2 + u_2(t) \\ &\quad + K_2(P, x_1, \hat{x}_2, t)(y(x_1, \hat{x}_2, t) - C_2(x_1, \hat{x}_2, t)\hat{x}_2) \end{aligned} \quad (5)$$

$$\begin{aligned} \dot{P} &= A(x_1, t)P + PA^\top(x_1, t) \\ &\quad - PC^\top(x_1, \hat{x}_2, t)Q(t)C(x_1, \hat{x}_2, t) + V(t) \end{aligned} \quad (6)$$

with $P(0)$ a symmetric positive definite matrix, Q and V bounded continuous symmetric positive semidefinite matrix-valued functions, and the "gain" K given by

$$\begin{aligned} K(P, x_1, \hat{x}_2, t) &= k(t)PC^\top(x_1, \hat{x}_2, t)Q(t) \\ K &= \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \end{aligned} \quad (7)$$

with $0.5 \leq k(t) \leq k_{max} < \infty$.

Remarks:

- One recognizes a CRE in (6) and may interpret (5) as a pre-observer of the state x_2 .
- If A is skew-symmetric a particular solution to this CRE, obtained by setting $V(t) = P(0)C^\top(x_1(t), \hat{x}_2(t), t)Q(t)C(x_1(t), \hat{x}_2(t), t)P(0)$ is the identity matrix $I_{2n \times 2n}$. In this case the gain K simplifies to $K = kC^\top Q$.
- By setting $\hat{x} = (\hat{x}_1^\top, \hat{x}_2^\top)^\top$ with $\hat{x}_1(t) = 0$ ($\forall t \geq 0$), (5) is equivalent to

$$\begin{aligned} \dot{\hat{x}} &= A(x_1, t)\hat{x} + \begin{pmatrix} u_1 \\ u_2(t) \end{pmatrix} \\ &\quad + K(P, x_1, \hat{x}_2, t)(y(x_1, \hat{x}_2, t) - C(x_1, \hat{x}_2, t)\hat{x}) \end{aligned} \quad (8)$$

with

$$u_1 = -K_1(P, x_1, \hat{x}_2, t)(y(x_1, \hat{x}_2, t) - C(x_1, \hat{x}_2, t)\hat{x})$$

This writing shows the formal kinship between (5)-(6) and a standard Riccati observer –formally defined as a deterministic generalization of a Kalman filter with

$k(t)$ not necessarily equal to one and matrices $Q^{-1}(t)$ and $V(t)$ not necessarily equal to noise covariance matrices—applied to a LTV system. In Section IV pose observers are derived with x_1 representing the vector part of a Rodrigues unit quaternion, and the kinship of these observers with so-called Multiplicative Extended Kalman Filters (MEKF) [13] will then appear clearly to the informed reader.

Theorem 3.1: Set

$$u_1 = -K_1(P, x_1, \hat{x}_2, t)(y(x_1, \hat{x}_2, t) - C_2(x_1, \hat{x}_2, t)\hat{x}_2) \quad (9)$$

and choose some positive definite matrix for $P(0)$. Define $A^*(t) := A(0, t)$, $C^*(t) := C(0, x_2(t), t)$, $\hat{x}_2 := x_2 - \hat{x}_2$. If there exist $\delta > 0$ and $\epsilon > 0$ such that $W_V^{A^*, C^*}(t, t + \delta) > \epsilon I_{2n}$ and $W_Q^{A^*, C^*}(t, t + \delta) > \epsilon I_{2n}$ for all $t > 0$, then $(x_1, \hat{x}_2) = (0, 0)$ is (locally) exponentially stable.

Corollary 3.2: If $Q(t)$ and $V(t)$ are larger than some positive matrix and the pair (A^*, C^*) is uniformly observable, then $(x_1, \hat{x}_2) = (0, 0)$ is (locally) exponentially stable.

Proof of Theorem 3.1

Define $\hat{x} = (\hat{x}_1^\top, \hat{x}_2^\top)^\top$ with $\hat{x}_1(t) = 0$ ($\forall t \geq 0$) so that (8) holds true with u_1 chosen as in the theorem. Define $\tilde{x} := x - \hat{x} = (x_1^\top, \hat{x}_2^\top)^\top$. Assume for the time being that $P(t)$, $P^{-1}(t)$ and $C(x_1(t), \hat{x}_2(t), t)$ are uniformly bounded along the solutions to the systems (3) and (5)-(6) w.r.t. initial conditions taken in a small neighborhood of $\tilde{x} = 0$. Using (4) and subtracting both members of (8) from the members of equality (3) yields

$$\dot{\tilde{x}} = (A(x_1, t) - K(P, x_1, \hat{x}_2, t)C(x_1, \hat{x}_2, t))\tilde{x} + O(|\tilde{x}|^2) \quad (10)$$

This relation shows that $\tilde{x} = 0$ is an equilibrium. One must now show that this equilibrium is attractive. Let us consider two cases depending on the choice of $V(t)$.

case 1: $V(t)$ is chosen larger than some positive definite matrix.

Define the positive function $L(\tilde{x}, t) := \tilde{x}^\top P^{-1}(t)\tilde{x}$. From this relation and the definition of K , and using the fact that $\frac{d}{dt}P^{-1} = -P^{-1}\dot{P}P^{-1}$, it comes that

$$\dot{L} = -\tilde{x}((2k-1)\tilde{x}^\top C^\top QC + P^{-1}VP^{-1})\tilde{x} + 2\tilde{x}^\top P^{-1}O(|\tilde{x}|^2)$$

Therefore, using the assumed boundedness of P^{-1}

$$\dot{L} = -\tilde{x}((2k-1)\tilde{x}^\top C^\top QC + P^{-1}VP^{-1})\tilde{x} + O(|\tilde{x}|^3)$$

with $O(|\tilde{x}|^3) = O(|L|^{3/2})$. Since $V(t)$ is larger than some positive definite matrix there exists $k_L > 0$ such that $\tilde{x}^\top P^{-1}VP^{-1}\tilde{x} \geq k_L L$. This yields

$$\dot{L} \leq -k_L L + O(|L|^{3/2}) \quad (11)$$

and, subsequently, the (local) exponential stability of $L = 0$. From the definition of L this in turn implies that $\tilde{x} = 0$ is locally exponentially stable.

There remains to prove that if $|\tilde{x}(0)|$ is small enough then $P(t)$, $P^{-1}(t)$ and $C(x_1(t), \hat{x}_2(t), t)$ are uniformly bounded. We already know that the boundedness of these matrices is

granted when $\tilde{x}(0) = 0$, as a consequence of the assumptions made on $W_V^{A^*, C^*}$ and $W_Q^{A^*, C^*}$.

Let $p_M(t)$ (resp. $p_m(t)$) denote the largest (resp. smallest) eigenvalue of $P(t)$. Since the matrices $A(x_1(t), t)$ and $V(t)$ are bounded, $p_M(t)$ cannot grow faster than exponentially with t , whereas $p_m(t)$ is always positive but may tend to zero (see Appendix of [1]). These properties of the CRE hold independently of $|\tilde{x}(0)|$ and $Q(t)$. Combining this maximum increase rate of $p_M(t)$ with the boundedness of $C^*(t)$ and the existence of $\gamma_c > 0$ such that $|C(x_1, \hat{x}_2, t) - C^*(t)| < \gamma_c |\tilde{x}|$ (by the assumed property of uniform continuity of $C(x_1, \hat{x}_2, t)$) one deduces, by inspection of the equation of evolution of the error \tilde{x} and the definition of the gain $K(P, x_1, \hat{x}_2, t)$, the existence of three positive exponentially increasing functions $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$ such that, given any $t_0 > \delta$

$$\frac{d}{dt}|\tilde{x}| \leq \beta_1(t_0)|\tilde{x}| + \beta_2(t_0)|\tilde{x}|^2 + \beta_3(t_0)|\tilde{x}|^3 ; t \in [0, t_0]$$

This relation in turn implies that given any $\epsilon_1 > 0$, there exists a positive number $\alpha(\epsilon_1, t_0)$ such that $|\tilde{x}(0)| < \alpha(\epsilon_1, t_0)$ implies that $|\tilde{x}(t)| \leq \epsilon_1$ for $t \in [0, t_0]$. Therefore, choosing ϵ_1 small enough and using again the uniform continuity of the matrix-valued functions A and C , ensures that $|W_V^A(t, t + \delta) - W_V^{A^*}(t, t + \delta)|$ and $|W_Q^{A, C}(t, t + \delta) - W_Q^{A^*, C^*}(t, t + \delta)|$ are smaller than $\epsilon/2$ for $t \in [0, t_0 - \delta]$. This in turn implies that $W_V^A(t, t + \delta) > \epsilon/2$ and $W_Q^{A, C}(t, t + \delta) > \epsilon/2$ for $t \in [0, t_0 - \delta]$ and, subsequently, that $p_M(t)$ (resp. $p_m(t)$) is in fact upper-bounded (resp. lower-bounded) on $[0, t_0]$ by a number $p_{M, \epsilon}$ (resp. $p_{m, \epsilon}$) that depends on ϵ , but not on t_0 . A lower bound of the gain k_L in (11), valid on the time interval $[0, t_0]$ when $|\tilde{x}(0)| < \alpha(\epsilon_1, t_0)$, is then equal to $\frac{p_{m, \epsilon}}{p_{M, \epsilon}^2} v_m$ with v_m denoting the (positive) ultimate lower bound of the eigenvalues of $V(t)$. From (11) we then deduce that $L(t) \leq L(0) \exp(-0.5 \frac{p_{m, \epsilon}}{p_{M, \epsilon}^2} v_m t)$ on the time interval $[0, t_0]$ provided that $\alpha(\epsilon_1, t_0)$ is chosen small enough. Since $L(0) \leq |\tilde{x}(0)|^2 / p_{m, \epsilon}$ and $L(t) \geq |\tilde{x}(t)|^2 / p_{M, \epsilon}$ on this time interval it also comes that $|\tilde{x}(t_0)| \leq \sqrt{\frac{p_{M, \epsilon}}{p_{m, \epsilon}} \exp(-0.5 \frac{p_{m, \epsilon}}{p_{M, \epsilon}^2} v_m t_0)} |\tilde{x}(0)|$. Therefore, there exists a value of t_0 such that $|\tilde{x}(t_0)| \leq |\tilde{x}(0)|$, provided that $\alpha(\epsilon_1, t_0)$ (the considered upper bound of $|\tilde{x}(0)|$) is chosen small enough. From there it only remains to repeat the same arguments with $\tilde{x}(t_0)$ taken as the new initial condition to establish that the previous bounds on $|\tilde{x}(t)|$ (and thus $|C(x_1(t), \hat{x}_2(t), t)|$) and on $P(t)$ continue to hold on the time interval $[t_0, 2t_0]$, provided that $|\tilde{x}(0)|$ is chosen small enough. Repeating this procedure for all time intervals $[jt_0, (j+1)t_0]$ ($j \in \mathbb{N}$) establishes the announced boundedness results on $[0, +\infty)$.

case 2:

$A(t)$ is skew-symmetric, $V(t) = P(t)C^\top(x_1(t), \hat{x}_2(t), t)Q(t)C(x_1(t), \hat{x}_2(t), t)P(t)$, and $P(t) = \begin{bmatrix} k_1 I_n & 0_n \\ 0_n & k_2 I_n \end{bmatrix}$, with $k_1 > 0$ and $k_2 > 0$, is the solution, used in (5), to the CRE (6).

In this case there exist skew-symmetric matrices $S_1(t)$ and $S_2(t)$ such that

$$A(x_1(t), t) = \begin{bmatrix} -S_1(t) & 0_n \\ 0_n & -S_2(t) \end{bmatrix}$$

Define the rotation matrices $R_1(t)$ and $R_2(t)$ in $SO(n)$ solutions to $\dot{R}_i = R_i S_i$ ($i = 1, 2$) with $R_1(0) = R_2(0) = I_n$. Then

$$\Phi(t, s) = \begin{bmatrix} R_1^\top(t) R_1(s) & 0_n \\ 0_n & R_2^\top(t) R_2(s) \end{bmatrix}$$

Define $\bar{x}(t) := \begin{bmatrix} R_1(t) \tilde{x}_1(t) / \sqrt{k_1} \\ R_2(t) \tilde{x}_2(t) / \sqrt{k_2} \end{bmatrix}$. From (10) and the fact that $KC = kPC^\top QC$ one gets

$$\begin{aligned} \dot{\bar{x}} &= -k \begin{bmatrix} \sqrt{k_1} I_n & 0_n \\ 0_n & \sqrt{k_2} I_n \end{bmatrix} B(t) \begin{bmatrix} \sqrt{k_1} I_n & 0_n \\ 0_n & \sqrt{k_2} I_n \end{bmatrix} \bar{x} \\ &\quad + 0(|\bar{x}|^2) \end{aligned} \quad (12)$$

with

$$\begin{aligned} B(t) &:= \bar{R}(t) C^\top (x_1(t), \hat{x}_2(t), t) Q(t) C (x_1(t), \hat{x}_2(t), t) \bar{R}^\top(t) \\ \bar{R}(t) &= \begin{bmatrix} R_1(t) & 0_n \\ 0_n & R_2(t) \end{bmatrix} \end{aligned}$$

Using the uniform continuity property of C and the boundedness of $Q(t)$, relation (12) also holds when replacing $B(t)$ by $B^*(t)$ with

$$B^*(t) := \bar{R}(t) C^{*\top}(t) Q(t) C^*(t) \bar{R}^\top(t)$$

By a slight generalization of Lemma 5 in [14] one deduces that $\bar{x} = 0$, and thus $\tilde{x} = 0$, are locally exponentially stable provided that there exist $\delta > 0$ and $\bar{\epsilon} > 0$ such that $\frac{1}{\delta} \int_t^{t+\delta} B^*(s) ds > \bar{\epsilon} I_{2n \times 2n}$ ($\forall t \geq 0$). This last inequality is clearly satisfied with $\bar{\epsilon} = \epsilon$ since

$$\frac{1}{\delta} \int_t^{t+\delta} B^*(s) ds = \bar{R}^\top(t) W_Q^{A^*, C^*}(t, t+\delta) \bar{R}(t)$$

Note that in this case one also has $W_Q^{A^*, C^*}(t, t+\delta) = W_V^{A^*}(t, t+\delta)$.

IV. APPLICATION TO POSE ESTIMATION FROM BEARING MEASUREMENTS

A. problem statement

Let us start with some notation:

- \mathcal{F} is an inertial frame.
- \mathcal{B} is a frame attached to a possibly moving body whose pose is estimated. Its origin is the point C . This point may, for instance, be the optic centre of a camera.
- $R(t)$ is the rotation matrix characterizing the orientation at time t of \mathcal{B} w.r.t. \mathcal{F} .
- $\omega(t)$ is the vector of coordinates at time t of the instantaneous angular velocity of \mathcal{B} expressed in the basis of \mathcal{B} , i.e. $\dot{R}(t) = R(t)S(\omega(t))$
- $p(t)$ (resp. $\bar{p}(t)$) is the vector of coordinates at time t of the position of C w.r.t. \mathcal{F} expressed in the basis of \mathcal{F} (resp. \mathcal{B}). Therefore $\bar{p} = R^\top p$.
- $v(t)$ (resp. $\bar{v}(t)$) is the vector of coordinates at time t of the velocity of C w.r.t. \mathcal{F} expressed in the basis of \mathcal{F} (resp. \mathcal{B}). Therefore: $\dot{p} = v$, $\bar{v} = R^\top v$, and $\dot{\bar{p}} = -S(\omega)\bar{p} + \bar{v}$.

We assume that the motion of \mathcal{B} keeps the distance $|p(t)|$ between the origins of \mathcal{F} and \mathcal{B} bounded.

- z_i ($i = 1, \dots, l$) is the vector of coordinates of the position of i th source point P_i w.r.t. \mathcal{F} expressed in the basis of this frame.
- $d_i(t) := R^\top(t)(p(t) - z_i)/|p(t) - z_i|$ ($i = 1, \dots, l$) is the unitary vector of coordinates characterizing the direction (or bearing) of $P_i C$ expressed in the basis of \mathcal{B} .
- $\hat{p}(t)$ (resp. $\hat{\bar{p}}(t)$) is an estimate at time t of $p(t)$ (resp. $\bar{p}(t)$).
- $\hat{R}(t)$ is an estimate at time t of $R(t)$. This is also a rotation matrix and the corresponding angular velocity vector expressed in the basis of \mathcal{B} is denoted as $\hat{\omega}(t)$, i.e. $\dot{\hat{R}} = \hat{R}S(\hat{\omega})$. Given an initial condition $\hat{R}(0)$ and $\hat{\omega}(t)$, $\hat{R}(t)$ can be calculated by numerical integration of the latter equality. This calculation may also be performed by using a unitary quaternion associated with $\hat{R}(t)$.

The problem at hand is to produce an on-line estimation of $p(t)$ and $R(t)$ given the knowledge of the l source point positions z_i , their directions $d_i(t)$, and either the inertial velocity vector $v(t)$ or the mobile velocity vector $\bar{v}(t)$. We will see that it is important to distinguish between these two velocity measurement cases in relation to observability conditions under which the estimated pose $(\hat{p}(t), \hat{R}(t))$ can robustly converge to the actual body pose $(p(t), R(t))$. We will first address the case the "mobile velocity measurement case" and then the more involved, but also more versatile, "inertial velocity measurement case".

Prior to deriving specific pose observers it is useful to recall a few facts about minimal parametrizations of the group of rotations, first order approximations of nonlinear functions and also kinematic relations repeatedly called for in the forthcoming developments.

Facts:

- There are infinitely many (local) minimal parametrizations of the three-dimensional Lie group $SO(3)$ of rotation matrices. Common ones are the vector part of a Hamilton or of a Rodrigues unit quaternion, Euler angles, Cardan (or Tait-Bryan) angles, etc. Theorem 3.1 can be adapted to any of these parametrizations to derive as many different pose observers endowed with similar local properties. We will favour here the vector part of a Rodrigues unit quaternion due to its simplicity of use and the fact that it provides a regular representation of rotation matrices for rotation angles up to π (excluded), by contrast with Euler angles that have representation singularities at $\pi/2$. The choice of quaternions lends itself to several possible parametrizations. We will consider two of them in this work.

The first one involves the unit quaternion $\Lambda = (\lambda_0, \lambda)$, with $\lambda \in \mathcal{B}_1^3$ (resp. $\lambda_0 \in \mathcal{B}_1^1$) denoting the vector (resp. scalar) part of the quaternion, associated with the "error" rotation matrix $\tilde{R} := R\hat{R}^\top$. Rodrigues formula

relating Λ to \tilde{R} is

$$\tilde{R} = I_3 + 2S(\lambda)(\lambda_0 I_3 + S(\lambda))$$

Define $\tilde{\omega} := \omega - \hat{\omega}$, we have the following kinematic relations

$$\dot{\tilde{R}} = \tilde{R}S(\hat{R}\tilde{\omega}) ; \quad \begin{cases} \dot{\lambda}_0 = -0.5(\hat{R}\tilde{\omega})^\top \lambda \\ \dot{\lambda} = 0.5\hat{R}\tilde{\omega}\lambda_0 + 0.5S(\hat{R}\tilde{\omega})\lambda \end{cases}$$

so that, using the fact that $\lambda_0 = 1 - |\lambda|^2$

$$\begin{cases} \dot{\tilde{R}} = \tilde{R}S(\lambda) + O(|\lambda|^2) \\ 2\dot{\lambda} = \hat{R}\tilde{\omega} + O(|\lambda||\tilde{\omega}|) \end{cases} \quad (13)$$

A second possible parametrization involves the unit quaternion $\bar{\Lambda} = (\bar{\lambda}_0, \bar{\lambda})$ associated with the "error" rotation matrix $\bar{\tilde{R}} := \tilde{R}^\top R$. Rodrigues formula relating $\bar{\Lambda}$ to $\bar{\tilde{R}}$ is

$$\bar{\tilde{R}} = I_3 + 2S(\bar{\lambda})(\bar{\lambda}_0 I_3 + S(\bar{\lambda}))$$

Define $\bar{\tilde{\omega}} := \omega - \bar{\tilde{R}}^\top \hat{\omega}$, we have the following kinematic relations

$$\dot{\bar{\tilde{R}}} = \bar{\tilde{R}}S(\bar{\tilde{\omega}}) ; \quad \begin{cases} \dot{\bar{\lambda}}_0 = -0.5\bar{\tilde{\omega}}^\top \bar{\lambda} \\ \dot{\bar{\lambda}} = 0.5\bar{\tilde{\omega}}\bar{\lambda}_0 + 0.5S(\bar{\tilde{\omega}})\bar{\lambda} \end{cases}$$

so that, using the the fact that $\bar{\lambda}_0 = 1 - |\bar{\lambda}|^2$

$$\begin{cases} \dot{\bar{\tilde{R}}} = \bar{\tilde{R}}S(\bar{\lambda}) + O(|\bar{\lambda}|^2) \\ 2\dot{\bar{\lambda}} = \bar{\tilde{\omega}} - S(\omega)(2\bar{\lambda}) + O(|\bar{\lambda}||\bar{\tilde{\omega}}|) + O(|\omega||\bar{\lambda}|^2) \end{cases} \quad (14)$$

Since $|\omega(t)|$ is bounded by assumption, one can also write

$$2\dot{\bar{\lambda}} = \bar{\tilde{\omega}} - S(\omega)(2\bar{\lambda}) + O(|\bar{\lambda}||\bar{\tilde{\omega}}|) + O(|\bar{\lambda}|^2)$$

- It is common knowledge that *first order* approximations of a nonlinear system $\dot{x} = f(x, t)$ about a current state-estimate \hat{x} can be used to derive state observers for this system. The classically invoked *linear* approximation $\dot{x} = \frac{\partial}{\partial x} f(\hat{x}, t)(x - \hat{x})$ is only one among infinitely many first order approximations. For instance any system $\dot{x} = \frac{\partial}{\partial x} f(\hat{x})(x - \hat{x}) + O(|x - \hat{x}|^2)$ is also a first order approximation about \hat{x} in the sense that the error committed in the right-hand side of the equality is still a $O(|x - \hat{x}|^2)$. This remark serves to point out that the observers derived further in the paper exploit first order approximations that are not necessarily linear approximations, and that several other variants of these observers can be obtained by just considering different first order approximations.

To complement these facts we recall that there are infinitely many ways to minimally parametrize the pose of a body in 3D-space and that each of these parametrizations yields specific linear and first-order approximations. The observer design methodology here considered can be adapted to any such parametrization and it can potentially produce infinitely many different observers having in common the same local stability and convergence properties. Let us for instance mention four parametrizations for which we will show how Theorem 3.1 applies to them in order to derive different pose Riccati observers, namely (λ, p) , $(\bar{\lambda}, p)$, (λ, \bar{p}) and $(\bar{\lambda}, \bar{p})$.

B. Mobile velocity measurement

From now on $R(t)$ is taken as a function of time so that $\hat{R} = \hat{R}^\top R = R\hat{R}$ can be seen either as a function of t and λ , or a function of t and $\bar{\lambda}$.

1) *Use of the parametrization (λ, p) :* In this case, using the first equation of (13)

$$\begin{aligned} \dot{p} &= v \\ &= R\bar{v} \\ &= \hat{R}\hat{R}\bar{v} \\ &= (\hat{R} - I_3)\hat{R}\bar{v} + \hat{R}\bar{v} \\ &= 2S(\lambda)\hat{R}\bar{v} + \hat{R}\bar{v} + O(|\bar{v}||\lambda|^2) \\ &= -S(\hat{R}\bar{v})(2\lambda) + \hat{R}\bar{v} + O(|\bar{v}||\lambda|^2) \end{aligned}$$

Setting $x_1 = 2\lambda$, $x_2 = p$ and using the second equation of (13), as well as the assumed boundedness of $|\omega(t)|$ and $|\bar{v}(t)|$, one obtains the system equation (3) with

$$A_{1,1} = A_{2,2} = 0_3, \quad A_{2,1}(x_1, t) = -S(\hat{R}\bar{v}(t)) \\ u_1 = \hat{R}\tilde{\omega}, \quad u_2 = \hat{R}\bar{v}$$

Concerning the output function y , one has for $i = 1, \dots, l$

$$\begin{aligned} |p - z_i|d_i &= R^\top(p - z_i) \\ &= \hat{R}^\top \hat{R}^\top(p - z_i) \\ &= \hat{R}^\top(p - z_i) + \hat{R}^\top(\hat{R}^\top - I_3)(p - z_i) \\ &= \hat{R}^\top(p - z_i) - \hat{R}^\top S(2\lambda)(p - z_i) \\ &\quad + O(|p - z_i||\lambda|^2) \\ &= \hat{R}^\top(p - z_i) + \hat{R}^\top S(\hat{p} - z_i)(2\lambda) \\ &\quad + O(|p - z_i||\lambda|^2) + O(|\lambda||\hat{p}|) \end{aligned}$$

so that, by setting $y_i(\lambda, p, t) := \Pi_{d_i(t)} \hat{R}^\top z_i$, using the identity $\Pi_{d_i} d_i = 0$ and the assumed boundedness of $|p(t) - z_i|$, one obtains

$$y_i(\lambda, p, t) = \Pi_{d_i(t)} \hat{R}^\top S(\hat{p} - z_i)(2\lambda) + \Pi_{d_i(t)} \hat{R}^\top p \\ + O(|\lambda|^2) + O(|\lambda||\hat{p}|)$$

Therefore, defining $y := (y_1^\top, \dots, y_l^\top)^\top$ one obtains the output equation (4) with

$$C_1(x_1, \hat{x}_2, t) = \begin{bmatrix} \Pi_{d_1(t)} \hat{R}^\top S(\hat{x}_2 - z_1) \\ \vdots \\ \Pi_{d_l(t)} \hat{R}^\top S(\hat{x}_2 - z_l) \end{bmatrix} \\ C_2(x_1, t) = \begin{bmatrix} \Pi_{d_1(t)} \hat{R}^\top \\ \vdots \\ \Pi_{d_l(t)} \hat{R}^\top \end{bmatrix}$$

The corresponding Riccati observer giving the expressions of $\hat{\omega}$ and \hat{p} (that are used to compute $\hat{R}(t)$ and $\hat{p}(t)$ via numerical integration) is then given by (5)-(7) and (9).

2) *Use of the parametrization $(\bar{\lambda}, p)$:* In this case, combining the first equation of (14) with the fact that $\dot{p} = R\bar{v} = \hat{R}\hat{R}\bar{v}$, one gets

$$\dot{p} = -\hat{R}S(\bar{v})(2\bar{\lambda}) + \hat{R}\bar{v} + O(|\bar{v}||\bar{\lambda}|^2)$$

Setting $x_1 = 2\bar{\lambda}$, $x_2 = p$ and using the second equation of (14), as well as the assumed boundedness of $|\omega(t)|$ and $|\bar{v}(t)|$, one obtains the system equation (3) with

$$A_{1,1}(t) = -S(\omega(t)), \quad A_{2,2} = 0_3, \quad A_{2,1}(x_1, t) = -\hat{R}S(\bar{v}(t)) \\ u_1 = \tilde{\omega}, \quad u_2 = \hat{R}\bar{v}$$

Concerning the output function y , one has for $i = 1, \dots, l$

$$\begin{aligned} |p - z_i|d_i &= R^\top(p - z_i) \\ &= \tilde{R}^\top \hat{R}^\top(p - z_i) \\ &= \hat{R}^\top(p - z_i) + (\tilde{R}^\top - I_3)\hat{R}^\top(p - z_i) \\ &= \hat{R}^\top(p - z_i) - S(2\bar{\lambda})\hat{R}^\top(p - z_i) \\ &\quad + O(|p - z_i||\bar{\lambda}|^2) \\ &= \hat{R}^\top(p - z_i) + S(\hat{R}^\top(\hat{p} - z_i))(2\bar{\lambda}) \\ &\quad + O(|p - z_i||\bar{\lambda}|^2) + O(|\bar{\lambda}||\bar{p}|) \end{aligned}$$

so that, by setting again $y_i(\lambda, p, t) := \Pi_{d_i(t)}\hat{R}^\top z_i$, using the identity $\Pi_{d_i}d_i = 0$ and the assumed boundedness of $|p(t) - z_i|$, one obtains

$$y_i(\bar{\lambda}, p, t) = \Pi_{d_i(t)}S(\hat{R}^\top(\hat{p} - z_i))(2\bar{\lambda}) + \Pi_{d_i(t)}\hat{R}^\top p + O(|\bar{\lambda}|^2) + O(|\bar{\lambda}||\bar{p}|)$$

Therefore, defining $y := (y_1^\top, \dots, y_l^\top)^\top$ one obtains the output equation (4) with

$$\begin{aligned} C_1(x_1, \hat{x}_2, t) &= \begin{bmatrix} \Pi_{d_1(t)}S(\hat{R}^\top(\hat{x}_2 - z_1)) \\ \vdots \\ \Pi_{d_l(t)}S(\hat{R}^\top(\hat{x}_2 - z_l)) \end{bmatrix} \\ C_2(x_1, t) &= \begin{bmatrix} \Pi_{d_1(t)}\hat{R}^\top \\ \vdots \\ \Pi_{d_l(t)}\hat{R}^\top \end{bmatrix} \end{aligned}$$

The corresponding Riccati observer giving the expressions of $\hat{\omega}$ and \hat{p} (that are used to compute $\hat{R}(t)$ and $\hat{p}(t)$ via numerical integration) is then given by (5)-(7) and (9).

3) *Use of the parametrization (λ, \bar{p}) :* In this case

$$\begin{cases} 2\dot{\lambda} &= \hat{R}\tilde{\omega} + O(|\lambda||\tilde{\omega}|) \\ \dot{\bar{p}} &= -S(\omega)\bar{p} + \bar{v} \end{cases}$$

Setting $x_1 = 2\lambda$, $x_2 = \bar{p}$ and using the assumed boundedness of $|\omega(t)|$ (note that boundedness of $|\bar{v}(t)|$ is no longer required at this level), one obtains the system equation (3) with

$$A_{1,1} = A_{2,1} = 0_3, \quad A_{2,2}(t) = -S(\omega(t)), \quad u_1 = \hat{R}\tilde{\omega}, \quad u_2 = \bar{v}$$

Concerning the output function y , one has for $i = 1, \dots, l$

$$\begin{aligned} |p - z_i|d_i &= \bar{p} - R^\top z_i \\ &= \bar{p} - \hat{R}^\top \hat{R}^\top z_i \\ &= \bar{p} - \hat{R}^\top z_i - \hat{R}^\top(\tilde{R}^\top - I_3)z_i \\ &= \bar{p} - \hat{R}^\top z_i + \hat{R}^\top S(2\lambda)z_i + O(|z_i||\lambda|^2) \\ &= \bar{p} - \hat{R}^\top z_i - \hat{R}^\top S(z_i)(2\lambda) + O(|z_i||\lambda|^2) \end{aligned}$$

so that, by setting again $y_i(\lambda, \bar{p}, t) := \Pi_{d_i(t)}\hat{R}^\top z_i$ and using the identity $\Pi_{d_i}d_i = 0$, one obtains

$$y_i(\lambda, \bar{p}, t) = -\Pi_{d_i(t)}\hat{R}^\top S(z_i)(2\lambda) + \Pi_{d_i(t)}\bar{p} + O(|\lambda|^2)$$

Therefore, defining $y := (y_1^\top, \dots, y_l^\top)^\top$ one obtains the output equation (4) with

$$\begin{aligned} C_1(x_1, \hat{x}_2, t) &= \begin{bmatrix} -\Pi_{d_1(t)}\hat{R}^\top S(z_1) \\ \vdots \\ -\Pi_{d_l(t)}\hat{R}^\top S(z_l) \end{bmatrix} \\ C_2(x_1, t) &= \begin{bmatrix} \Pi_{d_1(t)} \\ \vdots \\ \Pi_{d_l(t)} \end{bmatrix} \end{aligned}$$

The corresponding Riccati observer giving the expressions of $\hat{\omega}$ and \hat{p} (that are used to compute $\hat{R}(t)$ and $\hat{p}(t)$ via numerical integration) is then given by (5)-(7) and (9).

4) *Use of the parametrization $(\bar{\lambda}, \bar{p})$:* In this case

$$\begin{cases} 2\dot{\bar{\lambda}} &= -S(\omega)(2\bar{\lambda}) + \tilde{\omega} + O(|\bar{\lambda}||\tilde{\omega}|) + O(|\omega||\bar{\lambda}|^2) \\ \dot{\bar{p}} &= -S(\omega)\bar{p} + \bar{v} \end{cases}$$

Setting $x_1 = 2\bar{\lambda}$, $x_2 = \bar{p}$ and using the assumed boundedness of $|\omega(t)|$, one obtains the system equation (3) with

$$A_{1,1}(t) = A_{2,2}(t) = -S(\omega(t)), \quad A_{2,1} = 0_3, \quad u_1 = \tilde{\omega}, \quad u_2 = \bar{v}$$

Concerning the output function y , one has for $i = 1, \dots, l$

$$\begin{aligned} |p - z_i|d_i &= \bar{p} - R^\top z_i \\ &= \bar{p} - \tilde{R}^\top \hat{R}^\top z_i \\ &= \bar{p} - \hat{R}^\top z_i - (\tilde{R}^\top - I_3)\hat{R}^\top z_i \\ &= \bar{p} - \hat{R}^\top z_i + S(2\bar{\lambda})\hat{R}^\top z_i + O(|z_i||\bar{\lambda}|^2) \\ &= \bar{p} - \hat{R}^\top z_i - S(\hat{R}^\top z_i)(2\bar{\lambda}) + O(|z_i||\bar{\lambda}|^2) \end{aligned}$$

so that, by setting again $y_i(\bar{\lambda}, \bar{p}, t) := \Pi_{d_i(t)}\hat{R}^\top z_i$ and using the identity $\Pi_{d_i}d_i = 0$, one obtains

$$y_i(\bar{\lambda}, \bar{p}, t) = -\Pi_{d_i(t)}S(\hat{R}^\top z_i)(2\bar{\lambda}) + \Pi_{d_i(t)}\bar{p} + O(|\lambda|^2)$$

Therefore, defining $y := (y_1^\top, \dots, y_l^\top)^\top$ one obtains the output equation (4) with

$$\begin{aligned} C_1(x_1, \hat{x}_2, t) &= \begin{bmatrix} -\Pi_{d_1(t)}S(\hat{R}^\top z_1) \\ \vdots \\ -\Pi_{d_l(t)}S(\hat{R}^\top z_l) \end{bmatrix} \\ C_2(x_1, t) &= \begin{bmatrix} \Pi_{d_1(t)} \\ \vdots \\ \Pi_{d_l(t)} \end{bmatrix} \end{aligned}$$

The corresponding Riccati observer giving the expressions of $\hat{\omega}$ and \hat{p} (that are used to compute $\hat{R}(t)$ and $\hat{p}(t)$ via numerical integration) is then given by (5)-(7) and (9).

Remark:

The observers derived previously use the outputs $y_i = \Pi_{d_i}\hat{R}^\top z_i$. But other outputs may also be used to derive other observers. For instance, defining $\hat{d}_i := \hat{R}^\top(\hat{p} - z_i)/|\hat{p} - z_i|$ and using the fact that $\Pi_{\hat{d}_i}d_i = O(|\lambda|) + O(|\bar{p}|)$ provided that $|p(t) - z_i| > \mu > 0$ for all t , one has

$$\begin{aligned} |\hat{p} - z_i|\Pi_{\hat{d}_i}d_i &= |\hat{p} - z_i|\Pi_{\hat{d}_i}R^\top(p - z_i)/|p - z_i| \\ &= \Pi_{\hat{d}_i}R^\top(p - z_i) + \Pi_{\hat{d}_i}d_i(|\hat{p} - z_i| - |p - z_i|) \\ &= \Pi_{\hat{d}_i}R^\top(p - z_i) + O(|\lambda|^2 + |\bar{p}|^2) \end{aligned}$$

with

$$\begin{aligned} R^\top(p - z_i) &= \hat{R}^\top \hat{R}^\top(p - z_i) \\ &= \hat{R}^\top(p - z_i) + \hat{R}^\top S(\hat{p} - z_i)(2\lambda) \\ &\quad + O(|\lambda|^2) + O(|\lambda||\bar{p}|) \\ &= \hat{R}^\top(p - z_i) + |\hat{p} - z_i|S(\hat{d}_i)\hat{R}^\top(2\lambda) \\ &\quad + O(|\lambda|^2) + O(|\lambda||\bar{p}|) \end{aligned}$$

Therefore, setting $y_i := |\hat{p} - z_i|\Pi_{\hat{d}_i}d_i + \Pi_{\hat{d}_i}R^\top z_i$ yields $y_i = C_1(2\lambda) + C_2p + O(|\lambda|^2 + |\bar{p}|^2)$ with $C_1 = |\hat{p} - z_i|S(\hat{d}_i)\hat{R}^\top$ (using the fact that $\Pi_{x/|x|}S(x) = S(x)$) and $C_2 = \Pi_{\hat{d}_i}\hat{R}^\top$.

Similarly, defining $\hat{d}_i = (\hat{p} - \hat{R}^\top z_i) / |\hat{p} - \hat{R}^\top z_i|$ and using the fact that $\Pi_{\hat{d}_i} d_i = O(|\bar{\lambda}|) + O(|\bar{p}|)$ provided that $|p(t) - z_i| > \mu > 0$ for all t , one has

$$\begin{aligned} |\hat{p} - \hat{R}^\top z_i| \Pi_{\hat{d}_i} d_i &= |\hat{p} - \hat{R}^\top z_i| \Pi_{\hat{d}_i} (\bar{p} - R^\top z_i) / |\bar{p} - R^\top z_i| \\ &= \Pi_{\hat{d}_i} (\bar{p} - R^\top z_i) \\ &\quad + \Pi_{\hat{d}_i} d_i (|\hat{p} - \hat{R}^\top z_i| - |\bar{p} - R^\top z_i|) \\ &= \Pi_{\hat{d}_i} (\bar{p} - R^\top z_i) + O(|\bar{\lambda}|^2 + |\bar{p}|^2) \end{aligned}$$

with

$$\begin{aligned} \bar{p} - R^\top z_i &= \bar{p} - \bar{R}^\top \hat{R}^\top z_i \\ &= \bar{p} - \hat{R}^\top z_i - S(\hat{R}^\top z_i)(2\bar{\lambda}) + O(|\bar{\lambda}|^2) \end{aligned}$$

Therefore, setting $y_i := |\hat{p} - \hat{R}^\top z_i| \Pi_{\hat{d}_i} d_i + \Pi_{\hat{d}_i} \hat{R}^\top z_i$ yields $y_i = C_1(2\bar{\lambda}) + C_2\bar{p} + O(|\bar{\lambda}|^2 + |\bar{p}|^2)$ with $C_1 = -\Pi_{\hat{d}_i} S(\hat{R}^\top z_i)$ and $C_2 = \Pi_{\hat{d}_i}$.

As already mentioned all previously considered parametrizations yield Riccati observers sharing the same local properties. Determination and comparison of the associated domains of attraction in order to eventually work out to some type of efficiency ranking between these parametrizations are legitimate (but difficult) questions. They are not within the scope of the present work but they may motivate future studies. Note however that these issues are not necessarily critical in practice due to the existence, in a certain number of cases, of complementary algebraic solutions that can provide good initial estimates of the body pose [2].

5) *Observability issues:* Once the general expression of the observer is obtained it matters to determine conditions whose satisfaction ensures that the observer is exponentially stable, i.e. that zero estimation errors are uniformly exponentially stable. As pointed out in Theorem 3.1 and Corollary 3.2 uniform observability of the associated pair $(A^*(t), C^*(t))$ suffices, provided that the matrix $Q(t)$ entering the CRE is chosen larger than some positive matrix, and that the matrix $V(t)$ is either larger than some positive matrix or such that $W_V^{A^*}(t, t + \delta) > \epsilon I_6$. With $\Phi^*(t, s)$ denoting the transition matrix associated with $A^*(t)$, the strict positivity of the observability Grammian $W^*(t, t + \delta) := \int_t^{t+\delta} \Phi^{*\top}(s, t) C^{*\top}(s) C^*(s) \Phi^*(s, t) ds$ for some $\delta > 0$ and all $t \geq 0$ is thus the central property in this respect. When it is only semi-definite positive the condition number of the matrix $P(t)$ solution to the CRE generally diverges when the origin of the system $\dot{x} = A^*(t)x$ is not exponentially stable (as in the pose estimation case here considered) and the estimation errors do not uniformly exponentially converge to zero. We show next that the non-satisfaction of this property depends essentially upon the number of source points and, in the specific case of three source points, upon the body position relatively to the so-called "danger cylinder" well known of persons familiarized with the Perspective-n-Point (PnP) problem [2], [6].

One can verify that all observers considered so far share the same observability Grammian $W^*(t, t + \delta)$, modulo pre and post multiplication by regular matrices $B(t)$ and $B(t)^\top$ whose singular values are bounded from below and above by positive numbers and thus do not affect the uniform observability criterion. This was expected since all these observers

address the same estimation problem. Let us then consider, for example, the observer associated with the parametrization $(\bar{\lambda}, \bar{p})$ and examine the observability Grammian according to the number and disposition of the source points and the position/motion of the point C relatively to the source points. In the case of the parametrization $(\bar{\lambda}, \bar{p})$ one has

$$\begin{aligned} A^*(t) &= \begin{bmatrix} -S(\omega(t)) & 0_3 \\ 0_3 & -S(\omega(t)) \end{bmatrix} \\ C^*(t) &= \begin{bmatrix} -\Pi_{d_1(t)} S(R^\top(t) z_1) & \Pi_{d_1(t)} \\ \vdots & \vdots \\ -\Pi_{d_l(t)} S(R^\top(t) z_l) & \Pi_{d_l(t)} \end{bmatrix} \end{aligned}$$

and thus

$$\Phi^*(s, t) = \begin{bmatrix} R(s)^\top R(t) & 0_3 \\ 0_3 & R(s)^\top R(t) \end{bmatrix}$$

Define

$$\begin{aligned} \dot{R}(t) &:= \begin{bmatrix} R(t) & 0_3 \\ 0_3 & R(t) \end{bmatrix} \\ \Delta(s, t) &:= \Phi^{*\top}(s, t) C^{*\top}(s) C^*(s) \Phi^*(s, t) = \\ \dot{R}^\top(t) \sum_{i=1}^l \begin{bmatrix} S(z_i) \\ I_3 \end{bmatrix} \Pi_{d_i^F(s)} \begin{bmatrix} S^\top(z_i) & I_3 \end{bmatrix} \dot{R}(t) \end{aligned} \quad (15)$$

with $d_i^F(t) := (p(t) - z_i) / |p(t) - z_i|$ denoting the direction of the i th source point expressed in the basis of the inertial frame. The observability Grammian is thus never positive on any time interval if and only if there exists a non-zero vector $w \in \mathbb{R}^6$ such that

$$\Pi_{d_i^F(t)} \begin{bmatrix} S^\top(z_i) & I_3 \end{bmatrix} w = 0, \quad \forall i \in \{1, \dots, l\}, \quad \forall t \geq 0 \quad (16)$$

This equality characterizes body position trajectories along which the body pose is not uniformly observable on any time interval.

Aligned source points

Let us first consider the case of two source points, i.e. $i = 1, 2$. To satisfy the equality (16) it suffices that

$$\begin{bmatrix} S^\top(z_1) & I_3 \\ S^\top(z_2) & I_3 \end{bmatrix} w = 0 \quad (17)$$

One easily verifies that any vectors $w = \alpha \begin{bmatrix} z_2 - z_1 \\ z_1 \times z_2 \end{bmatrix}$, $\alpha \in \mathbb{R} - \{0\}$, is a solution to this equation. Moreover there is no other solution because the kernel of the matrix that w multiplies is one-dimensional.

By a simple extension of the previous proof, one shows that if all source points are aligned (independently of their number) then the observability Grammian is never positive whatever the position and motion of C .

Three non-aligned source points

In this case the satisfaction of (16) is equivalent to showing the existence of $(w, \dot{w}(t))$, with $\dot{w}(t) \in \mathbb{R}^3$ such that, for all $t \geq 0$

$$\begin{bmatrix} A_1 & A_2(t) \end{bmatrix} \begin{bmatrix} w \\ \dot{w}(t) \end{bmatrix} = 0 \quad (18)$$

with

$$A_1 := \begin{bmatrix} S^\top(z_1) & I_3 \\ S^\top(z_2) & I_3 \\ S^\top(z_3) & I_3 \end{bmatrix}$$

$$A_2(t) := \begin{bmatrix} p(t) - z_1 & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{3 \times 1} & p(t) - z_2 & 0_{3 \times 1} \\ 0_{3 \times 1} & 0_{3 \times 1} & p(t) - z_3 \end{bmatrix}$$

We already know that, if such a solution exists, $\dot{w}(t)$ cannot be identically equal to zero because the sub-matrix A_1 is of full rank equal to six. Now, forming the time derivative of both members of the equality (18) yields $\frac{d}{dt}(p(t) - z_i) = -(p(t) - z_i)(\frac{d}{dt}\dot{w}_i(t)/\dot{w}_i(t))$, $i = 1, 2, 3$, with \dot{w}_i denoting the i th component of \dot{w} . If the point C moves, i.e. $v(t) \neq 0$, the simultaneous satisfaction of these three equalities has implications namely i) the point C must move along a straight line, and ii) this line has to pass through one of the source points. Let us assume, for instance that this point is P_3 and that the direction of motion of C is given by a constant unit vector denoted as μ , i.e. $p(t) = z_3 + a(t)\mu$ with $a(t) \in \mathbb{R}$. Then the three equalities are satisfied with $\dot{w}_i(t) = \frac{d}{dt}\dot{w}_i(t) = 0$ for $i = 1, 2$, which in turn implies, in view of (18), that (17) is satisfied and thus that $w = \alpha \begin{bmatrix} z_2 - z_1 \\ z_1 \times z_2 \end{bmatrix}$. The third equation of (18) then yields $\alpha(z_3 \times z_2 + z_1 \times z_3 + z_1 \times z_2) + a(t)\dot{w}_3(t)\mu = 0$. This tells us that μ is orthogonal to the plane containing the three source points. Repeating the same argument for P_1 and P_2 one deduces that any motion of C along a straight line orthogonal to the source points plane and passing through a source point forbids uniform observability of the body pose.

Let us now examine the static case when C is motionless. The (9×9) matrix in the left-hand side of the equality (18) is then constant and one looks for p that renders this matrix singular. Via classical lines and columns manipulations one easily verifies that this matrix is singular if and only if the the following (6×6) matrix

$$D(p) := \begin{bmatrix} S^\top(z_2 - z_1) & z_1 - z_2 & p - z_2 & 0_{3 \times 1} \\ S^\top(z_3 - z_1) & z_1 - z_3 & 0_{3 \times 1} & p - z_3 \end{bmatrix}$$

is itself singular, i.e. if and only if $\det(D(p)) = 0$. We claim that this latter equation is nothing else than the equation of the so-called *danger cylinder* [2], [6], i.e. the circular cylinder generated by the circle passing through the three source points and whose axis is orthogonal to the plane containing the source points. Indeed, via a change of coordinates and scaling one can arbitrarily set $z_1 = [0, 0, 0]^\top$, $z_2 = [1, 0, 0]^\top$ and $z_3 = [a, b, 0]^\top$ ($b \neq 0$), and then easily verify that $\det(D(p)) = p_1^2 + p_2^2 - p_1 + (\frac{a(1-a)}{b} - b)p_2$, with p_i denoting the i th component of p .

To summarize, we have shown that the body pose is not uniformly observable when C is motionless on the danger cylinder, or when C moves along one of the three straight lines belonging to this cylinder and passing through a source point. If C moves on the danger cylinder, but

not along one of these three lines, then the body pose is uniformly observable under weak complementary conditions such as $\int_t^{t+\delta} |\Pi_\eta v(s)| ds > \epsilon > 0$, $\forall t \geq 0$, with η denoting a unit vector orthogonal to the source points plane. If C is fixed, but not on the danger cylinder, or moves without approaching the danger cylinder, or crosses this cylinder with a non-vanishing transversal velocity, uniform observability is also granted. Therefore, without minimizing the existence and practical significance of the particular trajectories of C for which the body pose is not observable, one may assert that uniform observability, and thus uniform exponential stability of the observers derived previously, are "generically" granted in the case of three non-aligned source points.

Four and more non-aligned source points

In the case of four source points (16) is equivalent to the existence of $(w, \dot{w}(t))$, with $w \in \mathbb{R}^6 - \{0\}$ and $\dot{w}(t) \in \mathbb{R}^4$, such that for all $t \geq 0$

$$\begin{bmatrix} A_1 & A_2(t) \end{bmatrix} \begin{bmatrix} w \\ \dot{w}(t) \end{bmatrix} = 0 \quad (19)$$

with

$$A_1 := \begin{bmatrix} S^\top(z_1) & I_3 \\ S^\top(z_2) & I_3 \\ S^\top(z_3) & I_3 \\ S^\top(z_4) & I_3 \end{bmatrix}$$

$$A_2(t) := \begin{bmatrix} p(t) - z_1 & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{3 \times 1} & p(t) - z_2 & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{3 \times 1} & 0_{3 \times 1} & p(t) - z_3 & 0_{3 \times 1} \\ 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & p(t) - z_4 \end{bmatrix}$$

By differentiating this equation w.r.t time and using the fact the sub-matrix A_1 is of full rank equal to six, one shows that this equation has no solution when C moves, i.e. when $v(t) \neq 0$ on some time interval.

If the point C is motionless one may arbitrarily choose it as the origin of the inertial frame, i.e. set $p = [0, 0, 0]^\top$. Relation (16) is then equivalent to requiring that all z_i ($i \in \{1, \dots, l\}$) are solutions to the equation

$$\Pi_{\frac{z}{|z|}} \begin{bmatrix} S(z) & I_3 \end{bmatrix} w = 0 \quad (20)$$

or, equivalently, are solutions to

$$z \times w_1 + \Pi_{\frac{z}{|z|}} w_2 = 0$$

for some $w = (w_1^\top, w_2^\top)^\top$. If $w_1 = [0, 0, 0]^\top$ and w_2 is different from zero, then this equation reduces to $\Pi_{\frac{z}{|z|}} w_2 = 0$. This implies that $z = \mu w_2$ ($\mu \in \mathbb{R}^*$). This is the equation of a straight line passing through the point C and one recovers the already established fact that the body pose is not uniformly observable when all source points are on a straight line containing the point C . If $w_2 = [0, 0, 0]^\top$ and w_1 is different from zero, then the equation reduces to $z \times w_1 = 0$ so that $z = \mu w_1$ ($\mu \in \mathbb{R}^*$). This is also the equation of a straight line passing through the point C and the same conclusion follows. A more interesting case is when neither w_1 nor w_2 are equal to zero. We show in the Appendix that in this case (20) is the equation of a family of *horopter*

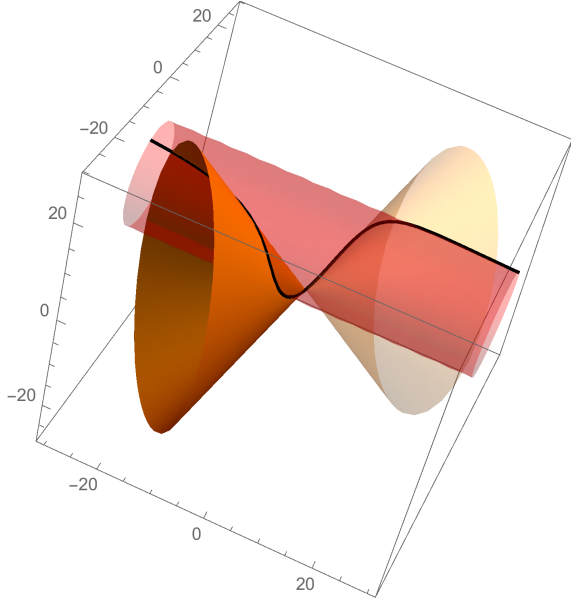


Fig. 1. Horopter curve at the intersection of a circular cylinder and an elliptic cone

curves well known in photogrammetry [15], [16]. We further show that every horopter i) lies on a cylinder, the so-called "dangerous cylinder of space resection", ii) lies also on an elliptic cone whose apex, the point with zero coordinates, is on the curve. The horopter curve is thus the intersection of these two surfaces (see Figure 1) and it passes through the cone's apex, the point C in the present case. This property does not seem to have been pointed out previously. Due to the specific role and location of the cone's apex on the curve we will refer to it as the horopter's origin. One can show that the horopter is uniquely determined by four given reference points. Therefore, if four source points are located on a horopter curve (which is then uniquely defined), and if C is the origin of the curve, then the body pose is not uniformly observable. Moreover, for a specific value of one of the horopter's parameters, the curve degenerates into a circle, perpendicular to the cylinder's axis, complemented with a straight line, parallel to the cylinder's axis, that intersects the circle at the point opposite to the horopter's origin w.r.t. the circle's centre. An even more degenerate case is when the circle shrinks to a point on the straight line, in which case the horopter is a straight line that passes through the origin. One deduces (this can also be verified directly from (20)) that the body pose is also not uniformly observable when all source points lie on a degenerate horopter. Such is the case when all source points are aligned without the point C being necessarily aligned with the source points (a case addressed previously), or when they are located on a circle containing also the point C , or when the source points are distributed on the circle and the straight line of a degenerate horopter whose origin is the point C . In all other cases the body pose is uniformly observable.

The same results holds for more than four source points.

C. Inertial velocity measurement

Alike the mobile velocity measurement case one can derive different Riccati observers by considering various parametrizations of the body pose and, in particular, by considering again the four parametrizations (λ, p) , $(\bar{\lambda}, p)$, (λ, \bar{p}) and $(\bar{\lambda}, \bar{p})$. Since all important relations and first order approximations involved in the design of the corresponding observers have already been derived, we will limit the exposition of this case to the sole parametrization (λ, p) .

1) *Use of the parametrization (λ, p) :* In this case $\dot{p} = v(t)$ with $v(t)$ being measured. Setting $x_1 = 2\lambda$, $x_2 = p$ and using the second equation of (13) one obtains the system equation (3) with

$$A = 0_6, \quad u_1 = \hat{R}\tilde{\omega}, \quad u_2 = v$$

Concerning the output function y , we have previously shown that, by setting $y_i(\lambda, p, t) := \Pi_{d_i(t)} \hat{R}^\top z_i$, using the identity $\Pi_{d_i} d_i = 0$ and the assumed boundedness of $|p(t) - \bar{p}|$, one has

$$y_i(\lambda, p, t) = \Pi_{d_i(t)} \hat{R}^\top S(\hat{p} - z_i)(2\lambda) + \Pi_{d_i(t)} \hat{R}^\top p + O(|\lambda|^2) + O(|\lambda||\bar{p}|)$$

so that, by defining $y := (y_1^\top, \dots, y_l^\top)^\top$, one obtains the output equation (4) with

$$C_1(x_1, \hat{x}_2, t) = \begin{bmatrix} \Pi_{d_1(t)} \hat{R}^\top S(\hat{x}_2 - z_1) \\ \vdots \\ \Pi_{d_l(t)} \hat{R}^\top S(\hat{x}_2 - z_l) \end{bmatrix}$$

$$C_2(x_1, t) = \begin{bmatrix} \Pi_{d_1(t)} \hat{R}^\top \\ \vdots \\ \Pi_{d_l(t)} \hat{R}^\top \end{bmatrix}$$

The corresponding Riccati observer giving the expressions of $\hat{\omega}$ and \hat{p} (that are used to compute $\hat{R}(t)$ and $\hat{p}(t)$ via numerical integration) is then given by (5)-(7) and (9).

2) *Observability issues:* From the expressions of A and C one deduces that $A^* = A = 0_6$, so that $\Phi^*(s, t) = I_6$, and

$$C^*(t) = \begin{bmatrix} \Pi_{d_1(t)} R^\top(t) S(p(t) - z_1) & \Pi_{d_1(t)} R^\top(t) \\ \vdots & \vdots \\ \Pi_{d_l(t)} R^\top(t) S(p(t) - z_l) & \Pi_{d_l(t)} R^\top(t) \end{bmatrix}$$

Therefore, in this case

$$\Delta(s, t) := \Phi^{*\top}(s, t) C^{*\top}(s) C^*(s) \Phi^*(s, t) = \sum_{i=1}^l \begin{bmatrix} S^\top(p(s) - z_i) \\ I_3 \end{bmatrix} \Pi_{d_i^F(s)} \begin{bmatrix} S(p(s) - z_i) & I_3 \end{bmatrix} \quad (21)$$

and the observability Grammian is never positive if there exists $w \in \mathbb{R}^6 - \{0\}$ such that

$$\Pi_{d_i^F(t)} \begin{bmatrix} S(p(t) - z_i) & I_3 \end{bmatrix} w = 0, \quad \forall i \in \{1, \dots, l\}, \quad \forall t \geq 0 \quad (22)$$

This relation is to be compared with the condition of non-uniform observability (16) associated with mobile velocity measurements. When the body position is constant, i.e. when $v(t) \equiv 0$, these two relations are equivalent.

To verify this one can use the change of coordinates $w = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 - p \times \bar{w}_1 \end{bmatrix}$. Then (22) is the same as (16) with w replaced by \bar{w} . This equivalence was expected since, in the zero velocity case, the observers derived by assuming either mobile or inertial velocity measurements coincide. However, there is an important difference in the case of body motion. Indeed, we have previously established that, in the mobile velocity measurement case, at least three source points are required to grant uniform observability of the body pose, whether the point C moves or is motionless. We show next that, when the inertial translational velocity of the body is measured and a single source point is used, the body pose is not uniformly observable only for very specific motions of the point C . In other words, body motion is better exploited for pose estimation in the sense that this estimation can be effectively performed with less information about the environment. Since the motionless case has been treated previously, only persistent C translational motion is now considered.

Single source point

To simplify, one may arbitrarily assume that the source point P coincides with the origin of the inertial frame, i.e. $z = [0, 0, 0]^\top$. Defining $d(t) := p(t)/|p(t)|$, the non-uniform observability condition (22) is then

$$\Pi_{d(t)} \begin{bmatrix} S(p(t)) & I_3 \end{bmatrix} w = 0, \quad \forall t \geq 0$$

or, equivalently, using the fact that $\Pi_{d(t)} S(p(t)) = S(p(t))$

$$p(t) \times w_1 + \Pi_{d(t)} w_2 = 0, \quad \forall t \geq 0 \quad (23)$$

with $w = (w_1^\top, w_2^\top)^\top$. This equation indicates that $p(t)$ must satisfy for all t the horopter equation (20) introduced previously, with the source point being now the origin of the horopter curve. The next proposition follows immediately:

Proposition 4.1: The body pose is not uniformly observable in the following situations

- 1) C moves along a horopter curve whose origin is the source point,
- 2) C moves along a straight line, or on a circle passing through the source point (the degenerate horopter's case).

Two and more source points

Since a non-degenerate horopter curve has a unique origin (that cannot coincide with two different source points), C motion along such a curve does not make the body pose non-observable uniformly. C motion on a circle passing through two source points does not forbid uniform stability either. Therefore non-observability occurs only when all source points and C are aligned. One can then show that, independently of the location of the source points, a sufficient condition for uniform observability is the existence of $\delta > 0$ and $\epsilon > 0$ such that $\forall t \geq 0 : \int_t^{t+\delta} v(s)v(s)^\top ds > \epsilon$. Note that this latter condition is not sufficient in the single source point case because the inequality can be satisfied by motions along a non-degenerate horopter.

V. SIMULATIONS

A. Three source points and mobile velocity measurements

This simulation illustrates that motion of C on the danger cylinder, by contrast with the motionless case, allows for uniform observability and effective pose estimation. All distances are expressed in meters, and we call the inertial plane $\{O; e_1, e_2\}$ the horizontal plane. The considered three source points are on this plane with coordinates respectively equal to $(0, 0, 0)$, $(5, 0, 0)$ and $(2.5, 2.5, 0)$. The circle passing through them is centred at $(2.5, 0, 0)$ and has a radius equal to 2.5. The point C moves on a circle parallel to it, ten meters above it and with a $1m/s$ translational velocity. Its coordinates are $p(t) = (2.5 + 2.5 \cos(0.4t), 2.5 \sin(0.4t), 10)^\top$ so that $v(t) = (-\sin(0.4t), \cos(0.4t), 0)$. The body's angular velocity is $\omega = (0.1 \sin(t), 0.4 \cos(2t), 0.6t)^\top \text{ rad/s}$. For the Riccati pose observer we choose the parametrization $(\bar{\lambda}, \bar{p})$ and use, to simplify, a diagonal matrix $Q = \text{diag}\{q_1 I_3, \dots, q_l I_3\}$. The observer equations are then

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\hat{p}} \end{bmatrix} &= \begin{bmatrix} \omega \\ -S(\omega)\hat{p} + \bar{v} \end{bmatrix} \\ &\quad - kP \begin{bmatrix} \sum_{i=1}^l q_i S(\hat{R}^\top z_i) \Pi_{d_i} (\hat{p} - \hat{R}^\top z_i) \\ \sum_{i=1}^l q_i \Pi_{d_i} (\hat{p} - \hat{R}^\top z_i) \end{bmatrix} \\ \dot{\hat{R}} &= \hat{R} S(\hat{\omega}) \\ \dot{\hat{P}} &= AP + PA^\top - PC^\top QCP + V \end{aligned} \quad (24)$$

with $k(t) \geq 0.5$ and

$$\begin{aligned} A &= \begin{bmatrix} -S(\omega) & 0_{3 \times 3} \\ 0_{3 \times 3} & -S(\omega) \end{bmatrix} \\ C &= \begin{bmatrix} -\Pi_{d_1} S(\hat{R}^\top z_1) & \Pi_{d_1} \\ \vdots & \vdots \\ -\Pi_{d_l} S(\hat{R}^\top z_l) & \Pi_{d_l} \end{bmatrix} \end{aligned}$$

with $l = 3$. For the reported simulation results we have taken $k = 1$ (as for a Kalman filter), $q_i = 10$ ($i = 1, 2, 3$), $V = \text{diag}\{0.1 I_3, I_3\}$ and $P(0) = \text{diag}\{I_3, 100 I_3\}$. Initial state values are $p(0) = (2.5, 0, 10)^\top$ and $R(0) = I_3$, whereas initial estimates are $\hat{p}(0) = (-2, 4, 3)^\top$ and $\bar{\Lambda}(0) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0)^\top$. This corresponds to an initial orientation angle error of 90 degrees.

Figures 2 (a) and (b) show the exponential convergence of the estimations errors to zero when the measurements of $\bar{v}(t)$, $\omega(t)$ and of the source points bearings are free of noise.

Figures 3 (a) and (b) show the time-evolution of the same estimation errors and illustrate the performance of the same observer in the case of noisy measurements. These results have been obtained by replacing $\bar{v}(t) = R^\top(t)v(t)$ and $\omega(t)$ respectively by $R^\top(t)(v(t) + b_v(t))$ and $\omega(t) + b_\omega(t)$ with b_v and b_ω denoting vectors of uncorrelated zero-mean Gaussian noises with standard deviations equal to 0.1 (for the components of b_v) and 0.01 (for the components of b_ω). As for the bearing measurements we have simulated noisy position measurements of the source points seen in the images of a calibrated camera by replacing d_i ($i = 1, 2, 3$)

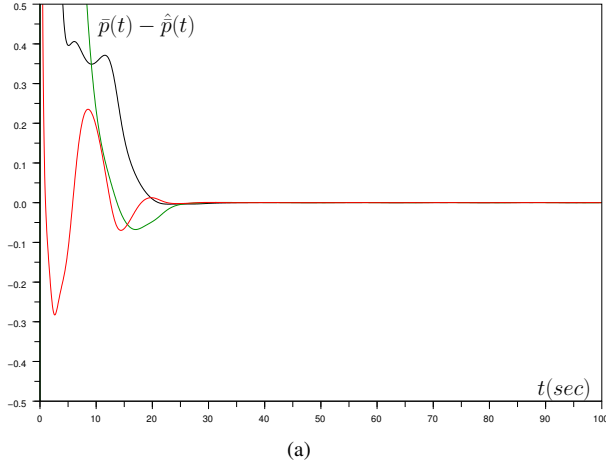


Fig. 2. Position error with noise-free measurements

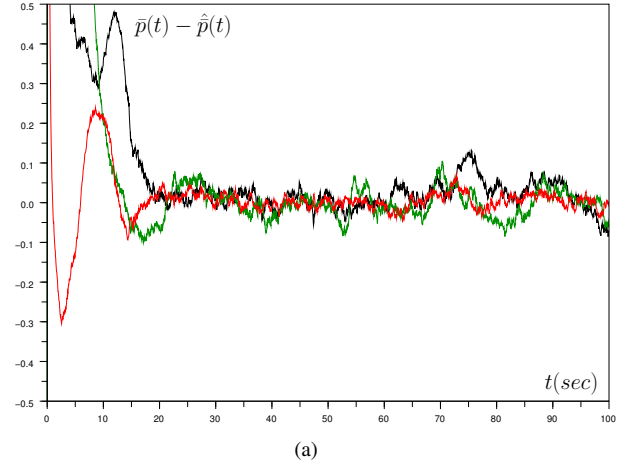


Fig. 3. Position error with noisy measurements

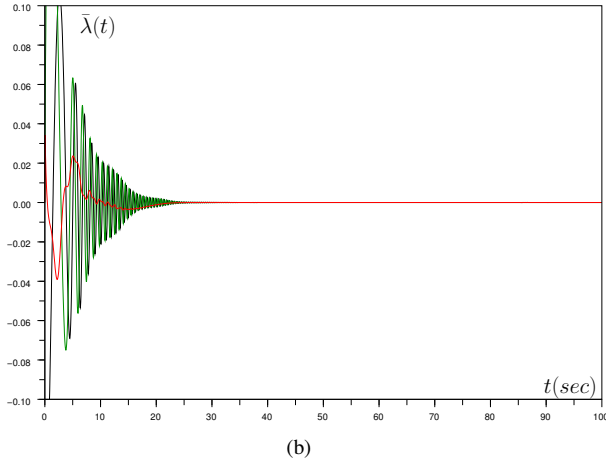


Fig. 2. Orientation error with noise-free measurements

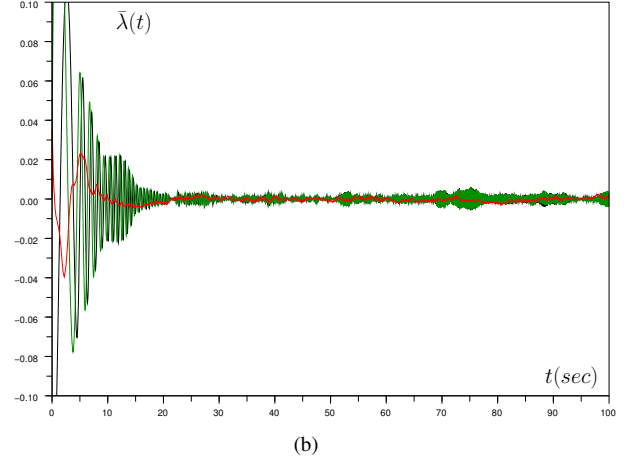


Fig. 3. Orientation error with noisy measurements

by

$$d_i^{mes} = \frac{\text{sign}(d_{i,3})}{\text{denom}} (d_{i,1}/d_{i,3} + n_{i,1}, d_{i,2}/d_{i,3} + n_{i,2}, 1)^\top$$

$$\text{denom} = \sqrt{(d_{i,1}/d_{i,3} + n_{i,1})^2 + (d_{i,2}/d_{i,3} + n_{i,2})^2 + 1}$$

with $n_{i,1}$ and $n_{i,2}$ denoting uncorrelated zero-mean uniformly distributed noises with maximum deviation equal to 0.005. For a CCD camera with an aperture of 90 degrees and producing images with (1000×1000) pixels this corresponds approximately to a maximum localisation error of 2.5 pixels, or 50cm in any direction at a distance of 10 meters. By comparison, the ultimate maximum estimation error of about 10cm observed in Figure 3 (a) illustrates the filtering property of the observer.

B. One source point and inertial velocity measurements

These simulations illustrate the possibility of estimating the body pose with bearing measurements of a single source point when the point C moves and its velocity is measured in the inertial frame. For the Riccati pose observer we choose the parametrization (λ, p) and use again, to simplify,

a diagonal matrix $Q = \text{diag}\{q_1 I_3, \dots, q_l I_3\}$. In the case of l source points, the observer equations are

$$\begin{aligned} \begin{bmatrix} \dot{\omega} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} \omega \\ v \end{bmatrix} \\ &- k \begin{bmatrix} \hat{R}^\top & 0_3 \\ 0_3 & I_3 \end{bmatrix} P \begin{bmatrix} \sum_{i=1}^l q_i \hat{R} S^\top(\xi_i) \Pi_{d_i} \xi_i \\ \sum_{i=1}^l q_i \hat{R} \Pi_{d_i} \xi_i \end{bmatrix} \\ \dot{\hat{R}} &= \hat{R} S(\omega) \\ \dot{P} &= -P C^\top Q C P + V \end{aligned} \quad (25)$$

with $\xi_i := \hat{R}^\top(\hat{p} - z_i)$, $k(t) \geq 0.5$, and

$$C = \begin{bmatrix} \Pi_{d_1} S(\xi_1) \hat{R}^\top & \Pi_{d_1} \hat{R}^\top \\ \vdots & \vdots \\ \Pi_{d_l} S(\xi_l) \hat{R}^\top & \Pi_{d_l} \hat{R}^\top \end{bmatrix}$$

In the present case, $l = 1$. The single source point used for pose estimation is the first of the three source points used in the previous simulations, i.e. $z_1 = [0, 0, 0]^\top$. The point C again moves on a danger cylinder passing through the source point, along a horizontal circular trajectory this time located 5 meters above the source point, and with a

translational velocity 2.5 times larger than in the previous simulations, i.e. $p(t) = [2.5 + 2.5 \cos(t), 2.5 \sin(t), 5]^\top$. The reason for these modifications (smaller distance to the source point, larger velocity) is to amplify the excitation properties associated with the motion of C in terms of observability, and subsequently increase the rate of convergence of the estimation errors to zero [1]. Initial pose estimates are $\hat{p}(0) = [3, 2, 7]^\top$ and $\Lambda(0) = [\sqrt{3}/2, 0.5, 0, 0]^\top$. This corresponds to an initial rotation angle error of 60 degrees. The initial value of the Riccati matrix is $P(0) = \text{diag}\{I_3, 10I_3\}$. The other parameters entering the Riccati equation (k , V and Q) and the measurement noises are the same as in the previous simulations.

Figures 4 (a) and (b) show the exponential convergence of the estimations errors to zero when the measurements of $v(t)$, $\omega(t)$ and of the source point bearing are free of noise.

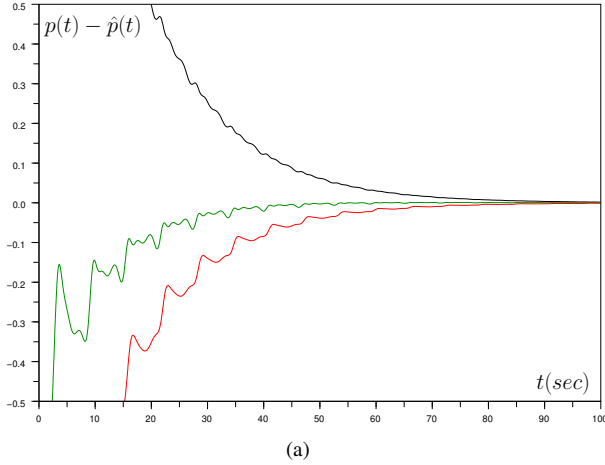


Fig. 4. Position error with noise-free measurements

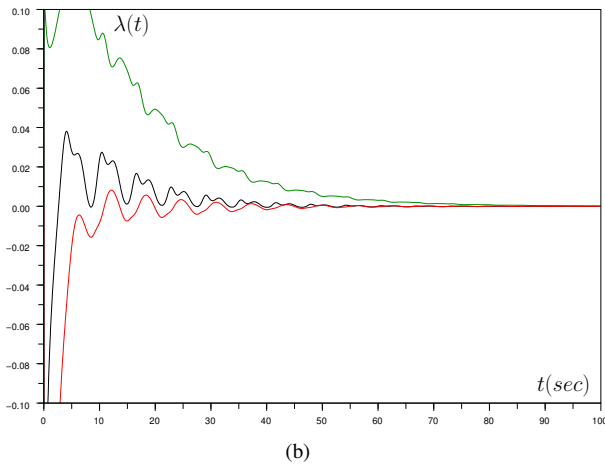


Fig. 4. Orientation error with noise-free measurements

Figures 5 (a) and (b) show the time-evolution of the same estimation errors and illustrate the performance of the observer in the case of noisy measurements.

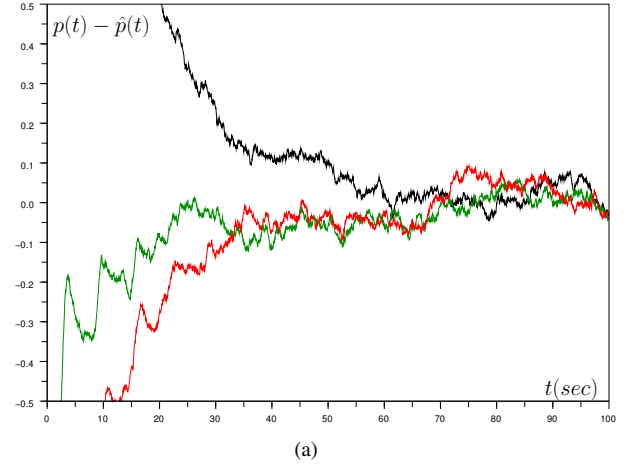


Fig. 5. Position error with noisy measurements

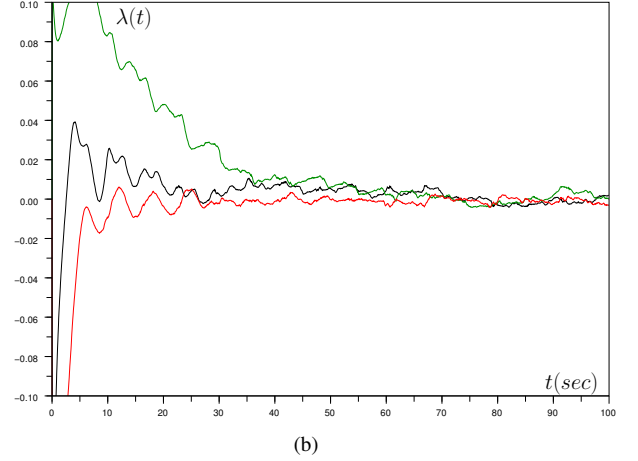


Fig. 5. Orientation error with noisy measurements

VI. STEREO VISION VS. MONOCULAR VISION

A stereo vision system consists in the rigid pairing of two (usually identical) cameras whose optic axes are parallel and orthogonal to the line joining the cameras' optic centres. Let the mid-point between the optic centres be the origin C of the mobile frame rigidly linked to the cameras, and a (resp. $-a$) the known vector of coordinates of one of the optic centres (resp. of the other optic centre) expressed in the mobile frame basis. Define also $p_{z_i} := R^\top(p - z_i)$, i.e. the opposite of the vector of coordinates of the i th source point expressed in the mobile frame basis, and denote the bearing of the i th source point measured from the image of the first (resp. second) camera by $d_{i,1}$ (resp. $d_{i,2}$), i.e. $d_{i,1} = \frac{p_{z_i} + a}{|p_{z_i} + a|}$ and $d_{i,2} = \frac{p_{z_i} - a}{|p_{z_i} - a|}$. Then $d_{i,1} \times d_{i,2} = \frac{2a \times p_{z_i}}{|p_{z_i} + a||p_{z_i} - a|}$, $a \times d_{i,1} = \frac{a \times p_{z_i}}{|p_{z_i} + a|}$ and $a \times d_{i,2} = \frac{a \times p_{z_i}}{|p_{z_i} - a|}$. Therefore $|p_{z_i} - a| = 2 \frac{|a \times d_{i,1}|}{|d_{i,1} \times d_{i,2}|}$ and $|p_{z_i} + a| = 2 \frac{|a \times d_{i,2}|}{|d_{i,1} \times d_{i,2}|}$. Since $p_{z_i} = d_{i,1}|p_{z_i} + a| - a$ (resp. $p_{z_i} = d_{i,2}|p_{z_i} - a| + a$) one deduces that $p_{z_i} = 2 \frac{d_{i,1}|a \times d_{i,2}|}{|d_{i,1} \times d_{i,2}|} - a$ (resp. $p_{z_i} = 2 \frac{d_{i,2}|a \times d_{i,1}|}{|d_{i,1} \times d_{i,2}|} + a$) and thus also

$$p_{z_i} = \frac{d_{i,1}|a \times d_{i,2}| + d_{i,2}|a \times d_{i,1}|}{|d_{i,1} \times d_{i,2}|} \quad (26)$$

This latter relation establishes the known fact that a stereo vision system allows for the measurement of the observed source point position expressed in the cameras frame, whereas a monocular vision system only allows for the source point bearing measurement. From there Riccati observers can be derived in the same way as for a single camera either by using source point(s) bearings measured in the images of the two cameras, without computing the p_{z_i} ($i = 1, \dots, l$) explicitly, or by using the p_{z_i} given by (26). In both cases one can verify that the non-uniform observability conditions (16) and (22) become

$$\begin{bmatrix} S^\top(z_i) & I_3 \end{bmatrix} w = 0, \quad \forall i \in \{1, \dots, l\}, \quad \forall t \geq 0 \quad (27)$$

for some $w \neq 0$, when the body translational velocity is measured in the mobile frame, and

$$\begin{bmatrix} S(p(t) - z_i) & I_3 \end{bmatrix} w = 0, \quad \forall i \in \{1, \dots, l\}, \quad \forall t \geq 0 \quad (28)$$

for some $w \neq 0$, when this velocity is measured in the inertial frame. These conditions are thus simply obtained by replacing the projection operators $\Pi_{d_i^F}$ by the identity matrix I_3 .

The first of these conditions tells us that, when the velocity of C is measured in the mobile frame, the body pose is not observable in the case of one and two source points, and that it is uniformly observable in the case of three or more non-aligned source points. These results are thus essentially the same as when using a monocular vision system, except for the non-existence of a danger cylinder in the three source points case and of specific singular body motions that do not grant uniform observability when three or more source points are used. However, a more significant difference in the case of three or more non-aligned source points, which results from the possibility of estimating the body attitude independently of the body position, is that almost global convergence of the pose estimation error can be proved despite the approximations made when deriving the equations of the Riccati observer (see, for instance, [17] where the identity matrix is implicitly used as a solution to the CRE associated with the attitude observer).

The second condition tells us that, in the case of persistent translational body motion, when the velocity is measured in the inertial frame and when a single source is used, the body pose is not uniformly observable when C moves along a straight line, but also that the satisfaction of $\forall t \geq 0 : \int_t^{t+\delta} v(s)v(s)^\top ds > \epsilon I_3$ for some δ and ϵ positive is sufficient to ensure uniform observability of the body pose. Conditions for uniform observability are thus again only slightly weaker than when using a monocular vision system.

VII. CONCLUDING REMARKS

Beyond the design and analysis of Riccati body pose observers from source points bearing measurements in the body frame, the present study points out the importance of body motion in the specific case when the body velocity is measured in an inertial frame. Indeed, when this velocity is measured in the mobile frame, body motion only marginally modifies the observability conditions under which pose estimation can be performed efficiently. Basically, at least three

source points whose positions are measured in the inertial frame are necessary in this case. By contrast, a single source point is generically sufficient when the body keeps moving and its velocity is measured in an inertial frame. Moreover, no measurement of the source point position is needed when the source point coincides with the origin of the chosen inertial frame w.r.t. which the body position is estimated. This latter fact enlightens the close kinship between the localization issue in the so-called Simultaneous-Localisation-and-Modelling (SLAM) problem based on source points bearing measurements and the pose estimation problem addressed in the present paper in the case of a single source point. In particular, it indicates that the localization problem is well-posed in terms of "consistency" (i.e. robust convergence of the pose estimation errors to zero) only when the body velocity is measured w.r.t. the chosen inertial frame. This is coherent with the fact that pose estimation w.r.t. an inertial frame is possible only when measurements made w.r.t. the inertial frame are available (the body velocity, in the present case). Measurements solely performed in the mobile frame (source points direction measurements and body velocity) cannot yield a consistent estimation. It seems that this simple but important fact has seldom, if ever, been clearly pointed out in the dedicated SLAM literature. This issue will be discussed more thoroughly in a forthcoming paper addressing the complete SLAM problem in the prolongation of the observer design and analysis methods here reported.

APPENDIX

A. Horopter's equations

We derive the horopter's equations from (20) when w_1 and w_2 are different from zero, and we show that the horopter may also be defined as the intersection of a circular cylinder with a cone asymptotically tangent to the cylinder.

Let e_i ($i = 1, 2, 3$) denote the canonical basis of \mathbb{R}^3 , i.e. $e_1 = (1, 0, 0)^\top$, $e_2 = (0, 1, 0)^\top$ and $e_3 = (0, 0, 1)^\top$. Since w_1 and w_2 are different from zero, one can arbitrarily set $w_1 = e_1$ and $w_2 = -(ke_1 + ae_2)$, with $a \in \mathbb{R}$ and $k \in \mathbb{R}$ being then the curve's parameters. Equation (20) then becomes

$$z \times e_1 - \Pi_{\frac{z}{|z|}}(ke_1 + ae_2) = 0$$

or, equivalently

$$|z|^2(z \times e_1) - (|z|^2 I_3 - zz^\top)((ke_1 + ae_2) = 0$$

Let z_i ($i = 1, 2, 3$) denote the coordinates of the vector z . Using these coordinates in the previous equality yields

$$\begin{aligned} -k|z|^2 + z_1(kz_1 + az_2) &= 0 \\ |z|^2(z_3 - a) + z_2(kz_1 + az_2) &= 0 \\ -|z|^2 z_2 + z_3(kz_1 + az_2) &= 0 \end{aligned}$$

therefore

$$\frac{kz_1 + az_2}{|z|^2} = \frac{k}{z_1} = -\frac{z_3 - a}{z_2} = \frac{z_2}{z_3} \quad (29)$$

so that, in view of the last equalities, $z_2^2 = z_3(a - z_3)$ and $z_1 = kz_3/z_2$. Assume for the time being that $a > 0$, then the positivity of z_2^2 implies that $z_3 \in [0, a]$ (≥ 0) and that $z_2 =$

$\pm\sqrt{z_3(a-z_3)}$. Therefore $z_1 = \pm k\sqrt{z_3}/\sqrt{a-z_3}$. It just remains to verify that these equalities are compatible with the first equality of (29), i.e. $-k|z|^2 + z_1(kz_1 + az_2) = 0$ or, equivalently, $k(z_2^2 + z_3^2) = az_1z_2$. In view of the expression of z_2 in terms of z_3 one has $k(z_2^2 + z_3^2) = akz_3$ and, using the expression of z_1 and z_2 in terms of z_3 one has $az_1z_2 = akz_3$. The expected equality thus holds true. The equations of the two curves obtained so far are

$$z_1 = k\sqrt{z_3}/\sqrt{a-z_3}, \quad z_2 = \sqrt{z_3(a-z_3)}, \quad z_3 \in [0, a] \quad (30)$$

and

$$z_1 = -k\sqrt{z_3}/\sqrt{a-z_3}, \quad z_2 = -\sqrt{z_3(a-z_3)}, \quad z_3 \in [0, a]$$

The first set of equations are the common equations of the horopter, and the second set provides the same geometrical curve rotated by an angle π . With $a < 0$ one also verifies that the same curves are again obtained modulo a symmetry w.r.t. the origin. Therefore all solutions to (20) yield the same geometrical horopter curve whose generic equations are given by (30).

The equality $z_2^2 + z_3^2 - az_3 = 0$ is satisfied by every point on the horopter. This is also the equation of a circular cylinder of diameter equal to a and whose axis direction is given by e_1 . Therefore the horopter lies on the surface of this cylinder. Another equality obtained previously is $k(z_2^2 + z_3^2) - az_1z_2 = 0$. This is the equation of an elliptic cone whose apex is the horopter's origin, whose central axis direction is given by the vector $[a, \sqrt{a^2 + k^2} - k, 0]^\top$ and whose geometric characteristic numbers are $d = (\sqrt{a^2 + k^2} - k)/(2k)$, $e = (\sqrt{a^2 + k^2} - k)/(\sqrt{a^2 + k^2} + k)$, and $f = 1$ (see figure 6). Therefore, the horopter curve lies also on the surface of this

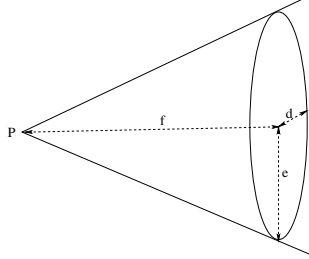


Fig. 6. Cone characteristic numbers

cone. The horopter curve is thus the intersection of a circular cylinder and an elliptic cone. Moreover, when z_3 tends to a , the coordinate z_1 tends to infinity. The horopter curve is thus unbounded and this implies that the cone is asymptotically tangent to the cylinder.

When the parameter k tends to zero the horopter degenerates into a circle of diameter a and a straight line perpendicular to the circle and intersecting it at the point opposite to the horopter's origin w.r.t. the circle's centre. When a tends to zero the horopter further degenerates into a straight line containing the origin.

B. Extension to the estimation of velocity biases

Velocity measurements are often corrupted by biases that are constant or slowly varying. In this case it is useful

to complement the observer with an estimation of these biases. An extension of the observer (25) is proposed next and its performance, when using a single source point and noisy measurements, is illustrated by simulation. The other Riccati observers considered in the present paper can easily be modified in the same manner.

Let ω_b denote the bias on the body's angular velocity so that $\frac{d}{dt}R(t) = R(t)(\omega(t) + \omega_b)$, with $\omega(t)$ the measured part of the angular velocity. Let similarly v_b denote a bias on the body's translational velocity so that $\dot{p}(t) = v(t) + v_b$, with $v(t)$ the measured part of the translational velocity. Via a straightforward extension of Theorem 3.1 involving different dimensions for x_1 and x_2 so as incorporate v_b and ω_b into the system's state vector, setting $\frac{d}{dt}\hat{R} = \hat{R}S(\hat{\omega} + \hat{\omega}_b)$ and using the approximation

$$\begin{aligned} 2\dot{\lambda} &= \hat{R}(\hat{\omega} + \hat{\omega}_b)\lambda_0 + 0.5S(\hat{R}(\hat{\omega} + \hat{\omega}_b))2\lambda \\ &= \hat{R}(\hat{\omega} + \hat{\omega}_b) + O(|\tilde{\omega}||\lambda|) + O(|\tilde{\omega}_b||\lambda|) \end{aligned}$$

one obtains the following equations of an extension of the Riccati observer (25)

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\hat{p}} \\ \dot{\hat{\omega}_b} \\ \dot{\hat{v}_b} \end{bmatrix} &= \begin{bmatrix} \omega \\ v + \hat{v}_b \\ 0_{3 \times 1} \\ 0_{3 \times 1} \end{bmatrix} \\ -k \begin{bmatrix} \hat{R}^\top & 0_{3 \times 9} \\ 0_{9 \times 3} & I_9 \end{bmatrix} P &\begin{bmatrix} \sum_{i=1}^l q_i \hat{R} S^\top(\xi_i) \Pi_{d_i} \xi_i \\ \sum_{i=1}^l q_i \hat{R} \Pi_{d_i} \xi_i \\ 0_{6 \times 1} \end{bmatrix} \\ \dot{P} &= AP + A^\top - PC^\top QCP + V \end{aligned} \quad (31)$$

with $\xi_i := \hat{R}^\top(\hat{p} - z_i)$, $k(t) \geq 0.5$,

$$A = \begin{bmatrix} 0_6 & \hat{R} & 0_3 \\ & 0_3 & I_3 \\ 0_6 & & 0_6 \end{bmatrix}$$

and

$$C = \begin{bmatrix} \Pi_{d_1} S(\xi_1) \hat{R}^\top & \Pi_{d_1} \hat{R}^\top \\ \vdots & \vdots \\ \Pi_{d_l} S(\xi_l) \hat{R}^\top & \Pi_{d_l} \hat{R}^\top \end{bmatrix}$$

For the simulation results reported next the same source point, translational and angular body motions, and measurement noises as for the case of unbiased velocities are used. The biases on the translational and angular velocities are $v_b = [0.1, -0.05, 0.2]^\top$ and $\omega_b = [0.01, 0.004, -0.02]^\top$ respectively. Figures (7) (a) and (b) show the time-evolution of the position and attitude estimation errors, whereas Figures (7) (c) and (d) show the time-evolution of the velocity biases estimation errors. The convergence of all estimation errors to small values illustrates the performance of the observer.

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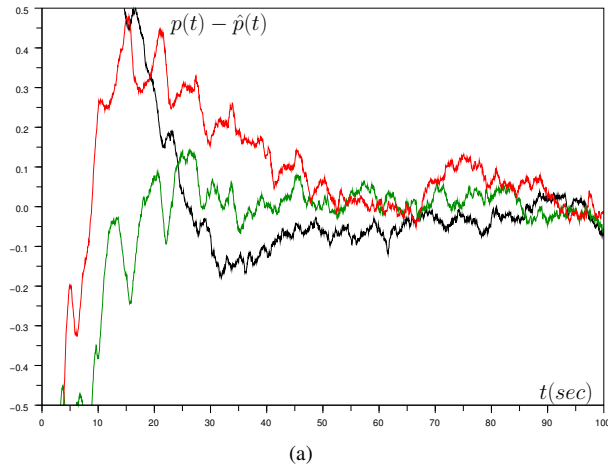


Fig. 7. Position error

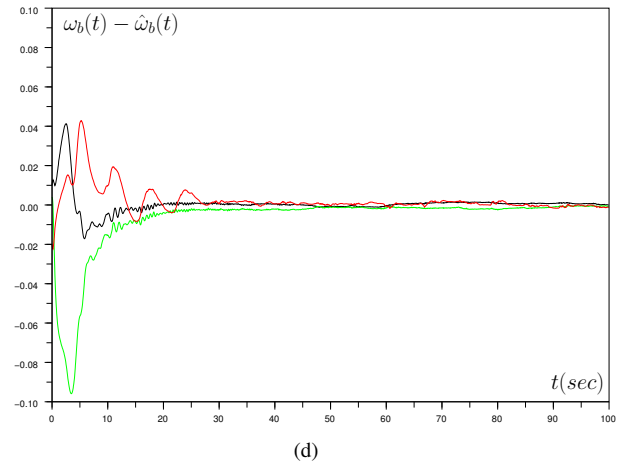


Fig. 7. Angular velocity bias error

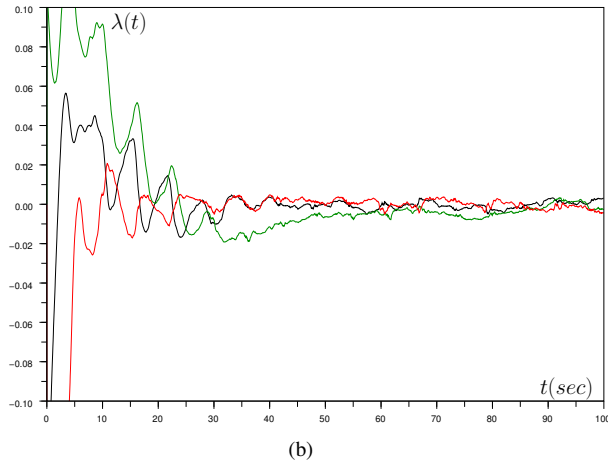


Fig. 7. Orientation error

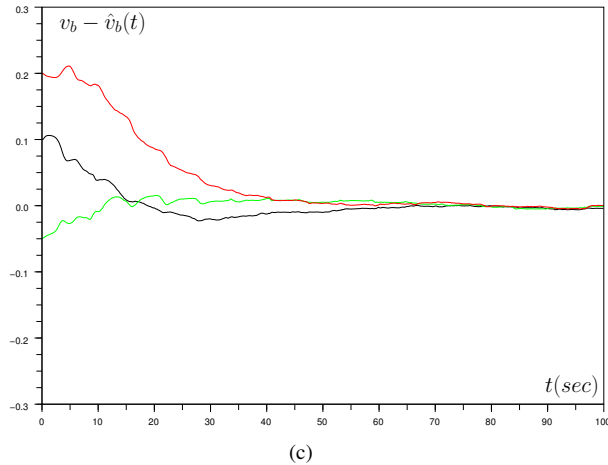


Fig. 7. Translational velocity bias error

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